Section 1.7

26. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$. The columns must linearly independent, by Theorem 7, because the first column is not zero, the second column is not a multiple of the first, and the third column is not a linear combination of the preceding two columns (because $\mathbf{a}_3$ is not in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$).

30. a. n

b. The columns of A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if $A\mathbf{x} = \mathbf{0}$ has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of A is a pivot column.

36. False. Counterexample: Take $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ all to be multiples of one vector. Take $\mathbf{v}_3$ to be *not* a multiple of that vector. For example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

38. True. If the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ had a nontrivial solution (with at least one of x_1, x_2, x_3 nonzero), then so would the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\cdot\mathbf{v}_4 = \mathbf{0}$. But that cannot happen because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be linearly independent. This problem can also be solved using Exercise 37, if you know that the statement there is true.

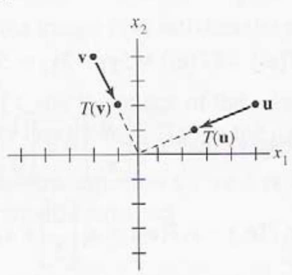
Section 1.8

10. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 & 0 \\ 0 & \textcircled{1} & 2 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

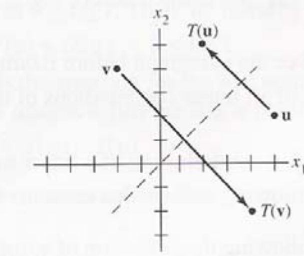
$$\begin{aligned} \textcircled{x_1} \quad + 3x_3 \quad &= 0 \\ \textcircled{x_2} + 2x_3 \quad &= 0 \\ \textcircled{x_4} &= 0 \end{aligned} \qquad \begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases} \qquad \mathbf{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

14.



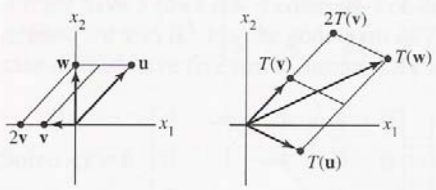
A contraction by the factor .5.

16.



A reflection through the line $x_2 = x_1$.

18. Draw a line through **w** parallel to **v**, and draw a line through **w** parallel to **u**. See the left part of the figure below. From this, estimate that $\mathbf{w} = \mathbf{u} + 2\mathbf{v}$. Since T is linear, $T(\mathbf{w}) = T(\mathbf{u}) + 2T(\mathbf{v})$. Locate $T(\mathbf{u})$ and $2T(\mathbf{v})$ as in the right part of the figure and form the associated parallelogram to locate $T(\mathbf{w})$.



24. Given any $\mathbf{x}$ in $\mathbf{R}^n$, there are constants $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p$, because $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbf{R}^n$. Then, from property (5) of a linear transformation,

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = c_1\mathbf{0} + \cdots + c_p\mathbf{0} = \mathbf{0}$$

30. Let $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for $\mathbf{x}$ in $\mathbf{R}^n$. If $\mathbf{b}$ is not zero, $T(\mathbf{0}) = A\mathbf{0} + \mathbf{b} = \mathbf{b} \neq \mathbf{0}$. Actually, T fails both properties of a linear transformation. For instance, $T(2\mathbf{x}) = A(2\mathbf{x}) + \mathbf{b} = 2A\mathbf{x} + \mathbf{b}$, which is not the same as $2T(\mathbf{x}) = 2(A\mathbf{x} + \mathbf{b}) = 2A\mathbf{x} + 2\mathbf{b}$. Also,

$$T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) + \mathbf{b} = A\mathbf{x} + A\mathbf{y} + \mathbf{b}$$

which is not the same as

$$T(\mathbf{x}) + T(\mathbf{y}) = A\mathbf{x} + \mathbf{b} + A\mathbf{y} + \mathbf{b}$$

36. Take **u** and **v** in $\mathbf{R}^3$ and let c and d be scalars. Then

$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$. The transformation T is linear because

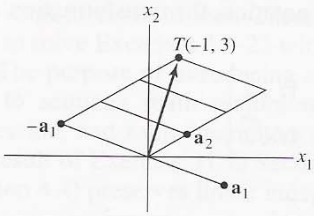
$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, 0, cu_3 + dv_3) = (cu_1, 0, cu_3) + (dv_1, 0, dv_3) \\ &= c(u_1, 0, u_3) + d(v_1, 0, v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$

Section 1.9

6. $T(\mathbf{e}_1) = \mathbf{e}_1$, $T(\mathbf{e}_2) = \mathbf{e}_2 + 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

10. $\mathbf{e}_1 \to -\mathbf{e}_1 \to -\mathbf{e}_2$ and $\mathbf{e}_2 \to \mathbf{e}_2 \to -\mathbf{e}_1$, so $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

14. Since $T(\mathbf{x}) = A\mathbf{x} = [\mathbf{a}_1 \;\; \mathbf{a}_2]\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = -\mathbf{a}_1 + 3\mathbf{a}_2$, when $\mathbf{x} = (-1, 3)$, the image of $\mathbf{x}$ is located by forming the parallelogram shown below.



18. As in Exercise 17, write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors. Since $\mathbf{x}$ has 2 entries, A has 2 columns. Since $T(\mathbf{x})$ has 4 entries, A has 4 rows.

$$\begin{bmatrix} 2x_2 - 3x_1 \\ x_1 - 4x_2 \\ 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} \\ & A & \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

28. The standard matrix A of the transformation T in Exercise 14 has linearly independent columns, because the figure in that exercise shows that $\mathbf{a}_1$ and $\mathbf{a}_2$ are not multiples. So T is one-to-one, by Theorem 12. Also, A must have a pivot in each column because the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. Thus, the echelon form of A is $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$. Since A has a pivot in each row, the columns of A span $\mathbf{R}^2$. So T maps $\mathbf{R}^2$ onto $\mathbf{R}^2$. An alternate argument for the second part is to observe directly from the figure in Exercise 14 that $\mathbf{a}_1$ and $\mathbf{a}_2$ span $\mathbf{R}^2$. This is more or less evident, based on experience with grids such as those in Figure 8 and Exercise 7 of Section 1.3.

30. By Theorem 12, the columns of the standard matrix A must span $\mathbb{R}^3$. By Theorem 4, the matrix must have a pivot in each row. There are four possibilities for the echelon form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Note that T cannot be one-to-one because of the shape of A.

32. The transformation T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$, by Theorem 12. This happens if and only if A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A has m rows, this happens if and only if A has m pivot columns. Thus, "T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only A has m pivot columns."