Section 2.2

12. If you assign this exercise, consider giving the following *Hint*: Use elementary matrices and imitate the proof of Theorem 7. The solution in the Instructor's Edition follows this hint. Here is another solution, based on the idea at the end of Section 2.2.

Write $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ and $X = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_p]$. By definition of matrix multiplication, $AX = [A\mathbf{u}_1 \ \cdots \ A\mathbf{u}_p]$. Thus, the equation $AX = B$ is equivalent to the p systems:

$$A\mathbf{u}_1 = \mathbf{b}_1, \ \ldots \ A\mathbf{u}_p = \mathbf{b}_p$$

Since A is the coefficient matrix in each system, these systems may be solved simultaneously, placing the augmented columns of these systems next to A to form $[A \ \ \mathbf{b}_1 \cdots \ \mathbf{b}_p] = [A \ \ B]$. Since A is invertible, the solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are uniquely determined, and $[A \ \ \mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$ must row reduce to $[I \ \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_p] = [I \ \ X]$. By Exercise 11, X is the unique solution $A^{-1}B$ of $AX = B$.

16. Let $C = AB$. Then $CB^{-1} = ABB^{-1}$, so $CB^{-1} = AI = A$. This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6.

20. **a.** Left-multiply both sides of $(A - AX)^{-1} = X^{-1}B$ by X to see that B is invertible because it is the product of invertible matrices.

b. Invert both sides of the original equation and use Theorem 6 about the inverse of a product (which applies because X^{-1} and B are invertible):

$$A - AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$$

Then $A = AX + B^{-1}X = (A + B^{-1})X$. The product $(A + B^{-1})X$ is invertible because A is invertible. Since X is known to be invertible, so is the other factor, $A + B^{-1}$, by Exercise 16 or by an argument similar to part (a). Finally,

$$(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X = X$$

24. If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^n$, then A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A is square, the pivots must be on the diagonal of A. It follows that A is row equivalent to I_n. By Theorem 7, A is invertible.

30. $$[A \quad I] = \begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4/5 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{bmatrix}. \quad A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$$

32. $$[A \quad I] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{bmatrix}.$$ The matrix A is not invertible.

Section 2.3

Section 2.3

4.

The matrix $\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$ obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible (or singular) by (e) in the IMT.

8.

The 4×4 matrix $\begin{bmatrix} 1 & 3 & 7 & 4 \\ 0 & 5 & 9 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ is invertible because it has four pivot positions, by (c) of the IMT.

12. **a.** True. If statement (k) of the IMT is true, then so is statement (j).

b. True. If statement (e) of the IMT is true, then so is statement (h).

c. True. See the remark immediately following the proof of the IMT.

d. False. The first part of the statement is not part (i) of the IMT. In fact, if A is any $n\times n$ matrix, the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ into $\mathbb{R}^n$, yet not every such matrix has n pivot positions.

e. True, by the IMT. If there is a **b** in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then statement (g) of the IMT is false, and hence statement (f) is also false. That is, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

14. If A is lower triangular with nonzero entries on the diagonal, then these n diagonal entries can be used as pivots to produce zeros below the diagonal. Thus A has n pivots and so is invertible, by the IMT. If one of the diagonal entries in A is zero, A will have fewer than n pivots and hence be singular.

Notes: For Exercise 14, another correct analysis of the case when A has nonzero diagonal entries is to apply the IMT (or Exercise 13) to A^T. Then use Theorem 6 in Section 2.2 to conclude that since A^T is invertible so is its transpose, A. You might mention this idea in class, but I recommend that you not spend much time discussing A^T and problems related to it, in order to keep from making this section too lengthy. (The transpose is treated infrequently in the text until Chapter 6.)

28.

Let W be the inverse of AB. Then $WAB = I$ and $(WA)B = I$. By (j) of the IMT applied to B in place of A, the matrix B is invertible.

34. The standard matrix of T is $A = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$, which is invertible because $\det A = 2 \neq 0$. By Theorem 9, T is invertible, and $T^{-1}(\mathbf{x}) = B\mathbf{x}$, where $B = A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$. Thus

$$T^{-1}(x_1, x_2) = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\frac{7}{2}x_1 + 4x_2, \frac{5}{2}x_1 + 3x_2\right)$$

Section 2.6

4. Solving as in Exercise 2:

$$\mathbf{d} = \mathbf{x} - C\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & .00 \\ .30 & .10 & .10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .9x_1 & -.6x_2 & -.6x_3 \\ -.3x_1 & +.8x_2 & \\ -.3x_1 & -.1x_2 & +.9x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \\ 0 \end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix} -.90 & -.60 & -.60 & 18 \\ -.30 & .80 & .00 & 18 \\ -.30 & -.10 & .90 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 73.33 \\ 0 & 1 & 0 & 50.00 \\ 0 & 0 & 1 & 30.00 \end{bmatrix}$$

so $\mathbf{x} = (73.33, 50.00, 30.00)$.

6. $\mathbf{x} = (I-C)^{-1}\mathbf{d} = \begin{bmatrix} .9 & -.6 \\ -.5 & .8 \end{bmatrix}^{-1}\begin{bmatrix} 18 \\ 11 \end{bmatrix} = \begin{bmatrix} 40/21 & 30/21 \\ 25/21 & 45/21 \end{bmatrix}\begin{bmatrix} 18 \\ 11 \end{bmatrix} = \begin{bmatrix} 50 \\ 45 \end{bmatrix}$

8. a. Given $(I-C)\mathbf{x} = \mathbf{d}$ and $(I-C)\Delta\mathbf{x} = \Delta\mathbf{d}$,

$$(I-C)(\mathbf{x} + \Delta\mathbf{x}) = (I-C)\mathbf{x} + (I-C)\Delta\mathbf{x} = \mathbf{d} + \Delta\mathbf{d}$$

Thus $\mathbf{x} + \Delta\mathbf{x}$ is the production level corresponding to a demand of $\mathbf{d} + \Delta\mathbf{d}$.

b. Since $\Delta\mathbf{x} = (I-C)^{-1}\Delta\mathbf{d}$ and $\Delta\mathbf{d}$ is the first column of I, $\Delta\mathbf{x}$ will be the first column of $(I-C)^{-1}$.

10. From Exercise 8, the (i, j) entry in $(I-C)^{-1}$ corresponds to the effect on production of sector i when the final demand for the output of sector j increases by one unit. Since these entries are all positive, an increase in the final demand for any sector will cause the production of all sectors to increase. Thus an increase in the demand for any sector will lead to an increase in the demand for all sectors.

Section 2.8

8. $[A \;\; \mathbf{p}] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & -2 & 0 & 1 \\ 0 & \textcircled{2} & -6 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Yes, the augmented matrix $[A \;\; \mathbf{p}]$ corresponds to a consistent system, so $\mathbf{p}$ is in Col A.

10. To determine whether $\mathbf{u}$ is in Nul A, simply compute $A\mathbf{u}$. Using A as in Exercise 7 and $\mathbf{u} = (-2, 3, 1)$,

$$A\mathbf{u} = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Yes, $\mathbf{u}$ is in Nul A.

$$A=\begin{bmatrix}-3&9&-2&-7\\2&-6&4&8\\3&-9&-2&2\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-3&6&9\\0&0&\textcircled{4}&5\\0&0&0&0\end{bmatrix}.\text{ Basis for Col }A:\ \begin{bmatrix}-3\\2\\3\end{bmatrix},\begin{bmatrix}-2\\4\\-2\end{bmatrix}.$$

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}\textcircled{1}&-3&0&1.50&0\\0&0&\textcircled{1}&1.25&0\\0&0&0&0&0\end{bmatrix}.\text{ This corresponds to: }\begin{aligned}\textcircled{x_1}-3x_2\quad+1.50x_4&=0\\\textcircled{x_3}+1.25x_4&=0.\\0&=0\end{aligned}$$

Solve for the basic variables and write the solution of $A\mathbf{x}=\mathbf{0}$ in parametric vector form:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3x_2-1.5x_4\\x_2\\-1.25x_4\\x_4\end{bmatrix}=x_2\begin{bmatrix}3\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-1.5\\0\\-1.25\\1\end{bmatrix}.\text{ Basis for Nul }A:\ \begin{bmatrix}3\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1.5\\0\\-1.25\\1\end{bmatrix}.$$

24.

26. $$A=\begin{bmatrix}3&-1&7&3&9\\-2&2&-2&7&5\\-5&9&3&3&4\\-2&6&6&3&7\end{bmatrix}\sim\begin{bmatrix}\textcircled{3}&-1&7&0&6\\0&\textcircled{2}&4&0&3\\0&0&0&\textcircled{1}&1\\0&0&0&0&0\end{bmatrix}.\text{ Basis for Col }A:\ \begin{bmatrix}3\\-2\\-5\\-2\end{bmatrix},\begin{bmatrix}-1\\2\\9\\6\end{bmatrix},\begin{bmatrix}3\\7\\3\\3\end{bmatrix}.$$

For Nul A,

$$[A\ \ \mathbf{0}]\sim\begin{bmatrix}\textcircled{1}&0&3&0&2.5&0\\0&\textcircled{1}&2&0&1.5&0\\0&0&0&\textcircled{1}&1&0\\0&0&0&0&0&0\end{bmatrix}\qquad\begin{aligned}\textcircled{x_1}+3x\quad+2.5x_5&=0\\\textcircled{x_2}+2x_3\quad+1.5x_5&=0\\\textcircled{x_4}+\quad x_5&=0\\0&=0\end{aligned}$$

The solution of $A\mathbf{x}=\mathbf{0}$ in parametric vector form:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-3x_3-2.5x_5\\-2x_3-1.5x_5\\x_3\\-x_5\\x_5\end{bmatrix}=x_3\underset{\mathbf{u}}{\begin{bmatrix}-3\\-2\\1\\0\\0\end{bmatrix}}+x_5\underset{\mathbf{v}}{\begin{bmatrix}-2.5\\-1.5\\0\\-1\\1\end{bmatrix}}.\text{ Basis for Nul }A:\{\mathbf{u},\mathbf{v}\}.$$

32.

If Nul R contains nonzero vectors, then the equation $R\mathbf{x}=\mathbf{0}$ has nontrivial solutions. Since R is square, the IMT shows that R is not invertible and the columns of R do not span $\mathbf{R}^6$. So Col R is a subspace of $\mathbf{R}^6$, but Col $R\neq\mathbf{R}^6$.

34.

If Nul $P=\{\mathbf{0}\}$, then the equation $P\mathbf{x}=\mathbf{0}$ has only the trivial solution. Since P is square, the IMT shows that P is invertible and the equation $P\mathbf{x}=\mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^5$. Also, each solution is unique, by Theorem 5 in Section 2.2.