6. $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}] = \begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 22 & 11 \\ 0 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}.$$

8. Fig. 2 suggests that $\mathbf{x} = 2\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{y} = 1.5\mathbf{b}_1 + \mathbf{b}_2$, and $\mathbf{z} = -\mathbf{b}_1 - .5\mathbf{b}_2$. If so, then

$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $[\mathbf{y}]_B = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$, and $[\mathbf{z}]_B = \begin{bmatrix} -1 \\ -.5 \end{bmatrix}$. To confirm $[\mathbf{y}]_B$ and $[\mathbf{z}]_B$, compute

$$1.5\mathbf{b}_1 + \mathbf{b}_2 = 1.5\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \mathbf{y} \text{ and } -\mathbf{b}_1 - .5\mathbf{b}_2 = -1\begin{bmatrix} 0 \\ 2 \end{bmatrix} - .5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix} = \mathbf{z}.$$

12. The information $A = \begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & -4 & 3 & 3 \\ 0 & 0 & \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{-5} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 3, and 5 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 5 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ -6 \end{bmatrix}$. For Nul A

$$[A \quad 0] \sim \begin{bmatrix} \textcircled{1} & 2 & 0 & -5 & 0 & 0 \\ 0 & 0 & \textcircled{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{aligned} \textcircled{x_1} + 2x_2 \quad - 5x_4 \quad &= 0 \\ \textcircled{x_3} - 2x_4 \quad &= 0 \\ \textcircled{x_5} &= 0 \end{aligned}$$

x_2 and x_4 are free variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

From this, dim Col A = 3 and dim Nul A = 2.

14. The five vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 3 & -6 & -9 & 15 \\ 0 & -4 & 8 & 12 & -20 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & 0 & -1 & 3 \\ 0 & \textcircled{-1} & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 2 of the original matrix form a basis for H, so $\dim H = 2$.

16. Col A *cannot* be $\mathbf{R}^3$ because the columns of A have four entries. (In fact, Col A is a 3-dimensional subspace of $\mathbf{R}^4$, because the 3 pivot columns of A form a basis for Col A.) Since A has 7 columns and 3 pivot columns, the equation $A\mathbf{x} = \mathbf{0}$ has 4 free variables. So, dim Nul $A = 4$.

22. The wording of this problem was poor in the first printing, because the phrase "it spans a four-dimensional subspace" was never defined. Here is a revision that I will put in later printings of the third edition:

Show that a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ in $\mathbf{R}^n$ is linearly dependent if dim Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_5\} = 4$.

Solution: Suppose that the subspace $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is four-dimensional. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ were linearly independent, it would be a basis for H. This is impossible, by the statement just before the definition of *dimension* in Section 2.9, which essentially says that *every* basis of a p-dimensional subspace consists of p vectors. Thus, $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ must be linearly dependent.

24. A rank 1 matrix has a one-dimensional column space. Every column is a multiple of some fixed To construct a 4×3 matrix, choose any nonzero vector in $\mathbf{R}^4$, and use it for one column. Choose multiples of the vector for the other two columns.

4. a. Let G stand for good weather, I for indifferent weather, and B for bad weather. Then the change in weather is given by the table

From: G	I	B	To:
.6	.4	.4	G
.3	.3	.5	I
.1	.3	.1	B

so the stochastic matrix is $P = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$.

b. The initial state vector is $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_1$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

Thus the chance of bad weather tomorrow is 20%.

c. The initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix} = \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix} = \begin{bmatrix} .48 \\ .336 \\ .184 \end{bmatrix}$$

Thus the chance of good weather on Wednesday is 48%.

10. Since $P^k = \begin{bmatrix} 1 & 1-.8^k \\ 0 & .8^k \end{bmatrix}$ will have a zero as its (2,1) entry for all k, so P is not a regular stochastic matrix.

14. From Exercise 4, $P = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.4 & .4 & .4 \\ .3 & -.7 & .5 \\ .1 & .3 & -.9 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix} -.4 & .4 & .4 & 0 \\ .3 & -.7 & .5 & 0 \\ .1 & .3 & -.9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ sum to 6, multiply by 1/6 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} \approx \begin{bmatrix} .5 \\ .333 \\ .167 \end{bmatrix}$. Thus in the long run the chance that a day has good weather is 50%.

16. [M] Let $P = \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.10 & .01 & .09 \\ .01 & -.10 & .01 \\ .09 & .09 & -.1 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing t

augmented matrix gives

$$\begin{bmatrix} -.10 & .01 & .09 & 0 \\ .01 & -.10 & .01 & 0 \\ .09 & .09 & -.1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.919192 & 0 \\ 0 & 1 & -.191919 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$ sum to

2.111111, multiply by 1/2.111111 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} .435407 \\ .090909 \\ .473684 \end{bmatrix}$. Thus on a typical da

about (.090909)(2000) = 182 cars will be rented or available from the downtown location.

18. If $\alpha = \beta = 0$ then $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that $P\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^2$, and that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two linearly independent steady-state vectors in this case.

If $\alpha \neq 0$ or $\beta \neq 0$, we solve $(P - I)\mathbf{x} = \mathbf{0}$ where $P - I = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$. Row reducing the augmented matrix gives

$$\begin{bmatrix} -\alpha & \beta & 0 \\ \alpha & -\beta & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\alpha x_1 = \beta x_2$, and one possible solution is to let $x_1 = \beta$, $x_2 = \alpha$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$. Since the entries in $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ sum to $\alpha + \beta$, multiply by $1/(\alpha + \beta)$ to obtain the steady-state vector $\mathbf{q} = \dfrac{1}{\alpha + \beta}\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$.

12. For $\lambda = 1$: $A - I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

The augmented matrix for $(A - I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-2/3)x_2$ and x_2 is free. A basis for the eigenspace corresponding to 1 is $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$.

The augmented matrix for $(A-5I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = 2x$

The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. A basis for the eigenspace is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

16. For $\lambda = 4$: $A-4I = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$[(A-4I) \quad \mathbf{0}] = \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So $x_1 = 2x_3, x_2 = 3x_3$, with x_3 and x_4 free variables. The general solution of $(A-4I)\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 3x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$ Basis for the eigenspace: $\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Note: I urge my students always to include the extra column of zeros when solving a homogeneous system. Exercise 16 provides a situation in which *failing* to add the column is likely to create problems for a student, because the matrix $A-4I$ itself has a column of zeros.

18. The eigenvalues of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are 4, 0, and -3, on the main diagonal, by Theorem 1.

20. The matrix $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of A. Eigenvectors for the eigenvalue 0 are solutions of $A\mathbf{x} = \mathbf{0}$ and therefore have entries that produce a linear dependence relation among the columns of A. Any nonzero vector (in $\mathbf{R}^3$) whose entries sum to 0 will work. Find any two such vectors that are not multiples; for instance, $(1, 1, -2)$ and $(1, -1, 0)$.

26. Suppose that A^2 is the zero matrix. If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{x}$ is nonzero, λ must be nonzero. Thus each eigenvalue of A is zero.

36. As in Exercise 35, $T(\mathbf{u}) = -\mathbf{u}$ and $T(\mathbf{v}) = 3\mathbf{v}$ because **u** and **v** are eigenvectors for the eigenvalu
-1 and 3, respectively, of the standard matrix A. Since T is linear, the image of **w** is
$T(\mathbf{w}) = T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.