Section 5.2

8. $A=\begin{bmatrix}7 & -2\\ 2 & 3\end{bmatrix}, A-\lambda I=\begin{bmatrix}7-\lambda & -2\\ 2 & 3-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(7-\lambda)(3-\lambda)-(-2)(2)=\lambda^2-10\lambda+25$$

Since $\lambda^2-10\lambda+25=(\lambda-5)^2$, the only eigenvalue is 5, with multiplicity 2.

10. $\det(A-\lambda I)=\det\begin{bmatrix}0-\lambda & 3 & 1\\ 3 & 0-\lambda & 2\\ 1 & 2 & 0-\lambda\end{bmatrix}$. From the special formula for 3×3 determinants, the characteristic polynomial is

$$\begin{aligned}\det(A-\lambda I)&=(-\lambda)(-\lambda)(-\lambda)+3\cdot 2\cdot 1+1\cdot 3\cdot 2-1\cdot(-\lambda)\cdot 1-2\cdot 2\cdot(-\lambda)-(-\lambda)\cdot 3\cdot 3\\ &=-\lambda^3+6+6+\lambda+4\lambda+9\lambda=-\lambda^3+14\lambda+12\end{aligned}$$

16. The determinant of a triangular matrix is the product of its diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & 0 & 0 & 0\\ 8 & -4-\lambda & 0 & 0\\ 0 & 7 & 1-\lambda & 0\\ 1 & -5 & 2 & 1-\lambda\end{bmatrix}=(5-\lambda)(-4-\lambda)(1-\lambda)^2$$

The eigenvalues are 5, 1, 1, and -4.

20.
$$\begin{aligned}\det(A^T-\lambda I)&=\det(A^T-\lambda I^T)\\ &=\det(A-\lambda I)^T && \text{Transpose property}\\ &=\det(A-\lambda I) && \text{Theorem 3(c)}\end{aligned}$$

24. First, observe that if P is invertible, then Theorem 3(b) shows that

$$1=\det I=\det(PP^{-1})=(\det P)(\det P^{-1})$$

Use Theorem 3(b) again when $A=PBP^{-1}$,

$$\det A=\det(PBP^{-1})=(\det P)(\det B)(\det P^{-1})=(\det B)(\det P)(\det P^{-1})=\det B$$

26. If $a\neq 0$, then $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\sim\begin{bmatrix}a & b\\ 0 & d-ca^{-1}b\end{bmatrix}=U$, and $\det A=(a)(d-ca^{-1}b)=ad-bc$. If $a=0$, then $A=\begin{bmatrix}0 & b\\ c & d\end{bmatrix}\sim\begin{bmatrix}c & d\\ 0 & b\end{bmatrix}=U$ (with one interchange), so $\det A=(-1)^1(cb)=0-bc=ad-bc$.

Section 5.3

2. $P=\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}, D=\begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$P^{-1}=\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}, D^4=\begin{bmatrix}1 & 0\\ 0 & 1/16\end{bmatrix}$, and $A^4=\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/16\end{bmatrix}\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}=\frac{1}{16}\begin{bmatrix}151 & 90\\ -225 & -134\end{bmatrix}$

4. $A^k = PD^kP^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4-3\cdot 2^k & 12\cdot 2^k-12 \\ 1-2^k & 4\cdot 2^k-3 \end{bmatrix}.$

6. As in Exercise 5, inspection of the factorization gives:

$$\lambda = 4: \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}; \lambda = 5: \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

14. The eigenvalues of A are given to be 5 and 4.

<u>For $\lambda = 5$</u>: $A-5I = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-5I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

<u>For $\lambda = 4$</u>: $A-4I = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-4I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Then set $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

20. Since A is triangular, its eigenvalues are obviously 4 and 2.

For $\lambda = 4$: $A-4I=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-2&0\\1&0&0&-2\end{bmatrix}$, and row reducing $[A-4I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1&0&0&-2&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}2\\0\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\\1\end{bmatrix}\right\}$.

For $\lambda = 2$: $A-2I=\begin{bmatrix}2&0&0&0\\0&2&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}$, and row reducing $[A-2I\ \ \mathbf{0}]$ yields $\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_3,\mathbf{v}_4\}=\left\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ and $\mathbf{v}_4$ construct $P=[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]=\begin{bmatrix}0&2&0&0\\1&0&0&0\\0&0&1&0\\0&1&0&1\end{bmatrix}$. Then set $D=\begin{bmatrix}4&0&0&0\\0&4&0&0\\0&0&2&0\\0&0&0&2\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

28. If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes,

$$\begin{aligned}A^T &= (PDP^{-1})^T = (P^{-1})^T D^T P^T\\ &= (P^T)^{-1}DP^T = QDQ^{-1}\end{aligned}$$

where $Q=(P^T)^{-1}$. Thus A^T is diagonalizable. By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.