

2. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$ are eigenvectors of a 3×3 matrix A , corresponding to

eigenvalues 3, $4/5$, and $3/5$, respectively. Also, $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. To describe the solution of the equation

$\mathbf{x}_{k+1} = A\mathbf{x}_k$ ($k = 1, 2, \dots$), first write \mathbf{x}_0 in terms of the eigenvectors.

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ -3 & -5 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow \mathbf{x}_0 = 2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$$

Then, $\mathbf{x}_1 = A(2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3) = 2A\mathbf{v}_1 + A\mathbf{v}_2 + 2A\mathbf{v}_3 = 2 \cdot 3\mathbf{v}_1 + (4/5)\mathbf{v}_2 + 2 \cdot (3/5)\mathbf{v}_3$. In general,

$\mathbf{x}_k = 2 \cdot 3^k \mathbf{v}_1 + (4/5)^k \mathbf{v}_2 + 2 \cdot (3/5)^k \mathbf{v}_3$. For all k sufficiently large,

$$\mathbf{x}_k \approx 2 \cdot 3^k \mathbf{v}_1 = 2 \cdot 3^k \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

4. $A = \begin{bmatrix} .5 & .4 \\ -.125 & 1.1 \end{bmatrix}$, $\det(A - \lambda I) = (.5 - \lambda)(1.1 - \lambda) - (.4)(.125) = \lambda^2 - 1.6\lambda + .6$. This characteristic polynomial factors as $(\lambda - 1)(\lambda - .6)$, so the eigenvalues are 1 and .6. For the eigenvalue 1, solve $(A - I)\mathbf{x} = 0$: $\begin{bmatrix} -.5 & .4 & 0 \\ -.125 & .1 & 0 \end{bmatrix} \sim \begin{bmatrix} -.5 & .4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. A basis for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Let \mathbf{v}_2 be an eigenvector for the eigenvalue .6. (The entries in \mathbf{v}_2 are not important for the long-term behavior of the system.) If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then $\mathbf{x}_1 = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2(.6)\mathbf{v}_2$, and for k sufficiently large,

$$\mathbf{x}_k = c_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + c_2 (.6)^k \mathbf{v}_2 \approx c_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Provided that $c_1 \neq 0$, the owl and wood rat populations each stabilize in size, and eventually the populations are in the ratio of 4 owls for each 5 thousand rats. If some aspect of the model were to change slightly, the characteristic equation would change slightly and the perturbed matrix A might not have 1 as an eigenvalue. If the eigenvalue becomes slightly large than 1, the two populations will grow; if the eigenvalue becomes slightly less than 1, both populations will decline.

6. When $p = .5$, $A = \begin{bmatrix} .4 & .3 \\ -.5 & 1.2 \end{bmatrix}$, and $\det(A - \lambda I) = \lambda^2 - 1.6\lambda + .63 = (\lambda - .9)(\lambda - .7)$.

The eigenvalues of A are .9 and .7, both less than 1 in magnitude. The origin is an attractor for the dynamical system and each trajectory tends toward $\mathbf{0}$. So both populations of owls and squirrels eventually perish.

The calculations in Exercise 4 (as well as those in Exercises 35 and 27 in Section 5.1) show that if the largest eigenvalue of A is 1, then in most cases the population vector \mathbf{x}_k will tend toward a multiple of the eigenvector corresponding to the eigenvalue 1. [If \mathbf{v}_1 and \mathbf{v}_2 are eigenvectors, with \mathbf{v}_1 corresponding to $\lambda = 1$, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then \mathbf{x}_k tends toward $c_1\mathbf{v}_1$, provided c_1 is not zero.] So the problem here is to determine the value of the predation parameter p such that the largest eigenvalue of A is 1. Compute the characteristic polynomial:

$$\det \begin{bmatrix} .4 - \lambda & .3 \\ -p & 1.2 - \lambda \end{bmatrix} = (.4 - \lambda)(1.2 - \lambda) + .3p = \lambda^2 - 1.6\lambda + (.48 + .3p)$$

By the quadratic formula,

$$\lambda = \frac{1.6 \pm \sqrt{1.6^2 - 4(.48 + .3p)}}{2}$$

The larger eigenvalue is 1 when

$$1.6 + \sqrt{1.6^2 - 4(.48 + .3p)} = 2 \text{ and } \sqrt{2.56 - 1.92 - 1.2p} = .4$$

In this case, $.64 - 1.2p = .16$, and $p = .4$.

$$18. \text{ a. } A = \begin{bmatrix} 0 & 0 & .42 \\ .6 & 0 & 0 \\ 0 & .75 & .95 \end{bmatrix}$$

$$\text{ b. } \text{ev} = \text{eig}(A) = \begin{bmatrix} 0.0774 + 0.4063i \\ 0.0774 - 0.4063i \\ 1.1048 \end{bmatrix}$$

The long-term growth rate is 1.105, about 10.5 % per year.

$$v = \text{nulbasis}(A - \text{ev}(3) * \text{eye}(3)) = \begin{bmatrix} 0.3801 \\ 0.2064 \\ 1.0000 \end{bmatrix}$$

For each 100 adults, there will be approximately 38 calves and 21 yearlings.

Note: The MATLAB box in the *Study Guide* and the various technology appendices all give directions for generating the sequence of points in a trajectory of a dynamical system. Details for producing a graphical representation of a trajectory are also given, with several options available in MATLAB, Maple, and Mathematica.