

Section 5.3

10. To find the eigenvalues of A , compute its characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{bmatrix} = (2 - \lambda)(1 - \lambda) - (3)(4) = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

Thus the eigenvalues of A are 5 and -2 .

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$. The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - x_2 = 0$, so $x_1 = x_2$ with x_2

free. The general solution is $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = -2$: $A + 2I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$. The equation $(A + 2I)\mathbf{x} = \mathbf{0}$ amounts to $4x_1 + 3x_2 = 0$, so $x_1 = (-3/4)x_2$

with x_2 free. The general solution is $x_2 \begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

From \mathbf{v}_1 and \mathbf{v}_2 construct $P = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$. Then set $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, where the eigenvalues in

D correspond to \mathbf{v}_1 and \mathbf{v}_2 respectively.

16. The eigenvalues of A are given to be 2 and 1.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$, and row reducing $[A - 2I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general

solution is $x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 1$: $A - I = \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix}$, and row reducing $[A - I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general

solution is $x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 construct $P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where

the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 respectively.

22. a. False. The n eigenvectors must be linearly independent. See the Diagonalization Theorem.
 b. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)
 c. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.
 d. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.
32. Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.

Section 6.6

2. The design matrix X and the observation vector \mathbf{y} are

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},$$

and one can compute

$$X^T X = \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix}, X^T \mathbf{y} = \begin{bmatrix} 6 \\ 25 \end{bmatrix}, \hat{\beta} = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} -.6 \\ .7 \end{bmatrix}$$

The least-squares line $y = \beta_0 + \beta_1 x$ is thus $y = -.6 + .7x$.

4. The design matrix X and the observation vector \mathbf{y} are

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

and one can compute

$$X^T X = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}, X^T \mathbf{y} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}, \hat{\beta} = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} 4.3 \\ -.7 \end{bmatrix}$$

The least-squares line $y = \beta_0 + \beta_1 x$ is thus $y = 4.3 - .7x$.

6. If the columns of X were linearly dependent, then the same dependence relation would hold for the vectors in \mathbb{R}^3 formed from the top three entries in each column. That is, the columns of the matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

would also be linearly dependent, and so this matrix (called a Vandermonde matrix)

would be noninvertible. Note that the determinant of this matrix is $(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \neq 0$ since x_1 , x_2 , and x_3 are distinct. Thus this matrix is invertible, which means that the columns of X are in fact linearly independent. By Theorem 14 in Section 6.5, the normal equations have a unique solution.

8. a. The model that produces the correct least-squares fit is $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where

$$X = \begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \text{ and } \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}$$

- b. [M] For the given data,

$$X = \begin{bmatrix} 4 & 16 & 64 \\ 6 & 36 & 216 \\ 8 & 64 & 512 \\ 10 & 100 & 1000 \\ 12 & 144 & 1728 \\ 14 & 196 & 2744 \\ 16 & 256 & 4096 \\ 18 & 324 & 5832 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 1.58 \\ 2.08 \\ 2.5 \\ 2.8 \\ 3.1 \\ 3.4 \\ 3.8 \\ 4.32 \end{bmatrix}$$

$$\text{so } \hat{\boldsymbol{\beta}} = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} .5132 \\ -.03348 \\ .001016 \end{bmatrix}, \text{ and the least-squares curve is } y = .5132x - .03348x^2 + .001016x^3.$$

18. Since

$$X^T X = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum x \\ \sum x & (\sum x)^2 \end{bmatrix}$$

$X^T X$ is a diagonal matrix when $\sum x = 0$.