Thus, a normal vector to the tangent plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\mathbf{i} - 4u\,\mathbf{j} + 4uv\,\mathbf{k}$$

Notice that the point (1, 1, 3) corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector there is

$$-2\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$$

Therefore, an equation of the tangent plane at (1, 1, 3) is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

or

$$x + 2y - 2z + 3 = 0$$

11.4 Exercises

1–4 ■ Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 4x^2 - y^2 + 2y$, $(-1, 2, 4)$

2. $z = e^{x^2 - y^2}$, $(1, -1, 1)$

3. $z = \sqrt{4 - x^2 - 2y^2}$, $(1, -1, 1)$

4. $z = y \ln x$, $(1, 4, 0)$

5–6 ■ Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

5. $z = x^2 + xy + 3y^2$, $(1, 1, 5)$

6. $z = \sqrt{x - y}$, $(5, 1, 2)$

CAS **7–8** ■ Draw the graph of f and its tangent plane at the given point. (Use your computer algebra system both to compute the partial derivatives and to graph the surface and its tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

7. $f(x, y) = e^{-(x^2+y^2)/15}(\sin^2 x + \cos^2 y)$, $(2, 3, f(2, 3))$

8. $f(x, y) = \dfrac{\sqrt{1 + 4x^2 + 4y^2}}{1 + x^4 + y^4}$, $(1, 1, 1)$

9–12 ■ Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

9. $f(x, y) = x\sqrt{y}$, $(1, 4)$

10. $f(x, y) = x/y$, $(6, 3)$

11. $f(x, y) = \tan^{-1}(x + 2y)$, $(1, 0)$

12. $f(x, y) = \sin(2x + 3y)$, $(-3, 2)$

13. Find the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at (2, 1) and use it to approximate $f(1.95, 1.08)$.

14. Find the linear approximation of the function $f(x, y) = \ln(x - 3y)$ at (7, 2) and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing f and the tangent plane.

15. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at (3, 2, 6) and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

16. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in the following table.

Duration (hours)

Wind speed (knots) v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

Use the table to find a linear approximation to the wave height function when v is near 40 knots and t is near 20 hours. Then estimate the wave heights when the wind has been blowing for 24 hours at 43 knots.

17. Use the table in Example 3 to find a linear approximation to the heat index function when the temperature is near 94 °F and the relative humidity is near 80%. Then estimate the heat index when the temperature is 95 °F and the relative humidity is 78%.

18. The wind-chill index I is the perceived temperature when the actual temperature is T and the wind speed is v, so we can write $I = f(T, v)$. The following table of values is an excerpt from a table compiled by the National Atmospheric and Oceanic Administration.

Wind speed (km/h)

Actual temperature (°C) T \ v	10	20	30	40	50
20	18	16	14	13	13
16	14	11	9	7	7
12	9	5	3	1	0
8	5	0	−3	−5	−6

Use the table to find a linear approximation to the wind chill index function when T is near 16 °C and v is near 30 km/h. Then estimate the wind chill index when the temperature is 14 °C and the wind speed is 27 km/h.

19–22 ■ Find the differential of the function.

19. $u = e^t \sin\theta$

20. $v = y \cos xy$

21. $w = \ln\sqrt{x^2 + y^2 + z^2}$

22. $u = r/(s + 2t)$

23. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

24. If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

25. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

26. The dimensions of a closed rectangular box are measured as 80 cm, 60 cm, and 50 cm, respectively, with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.

27. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

28. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

29. A boundary stripe 3 in. wide is painted around a rectangle whose dimensions are 100 ft by 200 ft. Use differentials to approximate the number of square feet of paint in the stripe.

30. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

31. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$, and $R_3 = 50\ \Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

32. Four positive numbers, each less than 50, are rounded to the first decimal place and then multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from the rounding.

33–36 ■ Find an equation of the tangent plane to the given parametric surface at the specified point. Use a computer to graph the surface and the tangent plane.

33. $x = u + v,\quad y = 3u^2,\quad z = u - v;\qquad (2, 3, 0)$

34. $x = u^2,\quad y = u - v^2,\quad z = v^2;\qquad (1, 0, 1)$

35. $\mathbf{r}(u, v) = uv\,\mathbf{i} + ue^v\,\mathbf{j} + ve^u\,\mathbf{k};\qquad (0, 0, 0)$

36. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + u\cos v\,\mathbf{j} + v\sin u\,\mathbf{k};\qquad (1, 1, 0)$

37–38 ■ Show that the function is differentiable by finding values of ε_1 and ε_2 that satisfy Definition 7.

37. $f(x, y) = x^2 + y^2$

38. $f(x, y) = xy - 5y^2$

39. Prove that if f is a function of two variables that is differentiable at (a, b), then f is continuous at (a, b). [*Hint:* Show that $\lim_{(\Delta x, \Delta y) \to (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$.]

40. (a) The function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

was graphed in Figure 4. Show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but f is not differentiable at $(0, 0)$. [*Hint:* Use the result of Exercise 39.]

(b) Explain why f_x and f_y are not continuous at $(0, 0)$.