

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid. If  $F$  is defined within a sphere containing  $(a, b, c)$ , where  $F(a, b, c) = 0$ ,  $F_z(a, b, c) \neq 0$ , and  $F_x, F_y,$  and  $F_z$  are continuous inside the sphere, then the equation  $F(x, y, z) = 0$  defines  $z$  as a function of  $x$  and  $y$  near the point  $(a, b, c)$  and the partial derivatives of this function are given by (7).

**EXAMPLE 9** Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if  $x^3 + y^3 + z^3 + 6xyz = 1$ .

**SOLUTION** Let  $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$ . Then, from Equations 7, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

▲ The solution to Example 9 should be compared to the one in Example 4 in Section 11.3.

**11.5**

**Exercises**

1-4 ■ Use the Chain Rule to find  $dz/dt$  or  $dw/dt$ .

- 1.  $z = \sin x \cos y, \quad x = \pi t, \quad y = \sqrt{t}$
- 2.  $z = x \ln(x + 2y), \quad x = \sin t, \quad y = \cos t$
- 3.  $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$
- 4.  $w = xy + yz^2, \quad x = e^t, \quad y = e^t \sin t, \quad z = e^t \cos t$

5-8 ■ Use the Chain Rule to find  $\partial z/\partial s$  and  $\partial z/\partial t$ .

- 5.  $z = x^2 + xy + y^2, \quad x = s + t, \quad y = st$
- 6.  $z = x/y, \quad x = se^t, \quad y = 1 + se^{-t}$
- 7.  $z = e^r \cos \theta, \quad r = st, \quad \theta = \sqrt{s^2 + t^2}$
- 8.  $z = \sin \alpha \tan \beta, \quad \alpha = 3s + t, \quad \beta = s - t$

9. If  $z = f(x, y)$ , where  $x = g(t), y = h(t), g(3) = 2, g'(3) = 5, h(3) = 7, h'(3) = -4, f_x(2, 7) = 6,$  and  $f_y(2, 7) = -8,$  find  $dz/dt$  when  $t = 3$ .

10. Let  $W(s, t) = F(u(s, t), v(s, t))$ , where  $u(1, 0) = 2, u_s(1, 0) = -2, u_t(1, 0) = 6, v(1, 0) = 3, v_s(1, 0) = 5, v_t(1, 0) = 4, F_u(2, 3) = -1,$  and  $F_v(2, 3) = 10.$  Find  $W_s(1, 0)$  and  $W_t(1, 0).$

11-14 ■ Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

- 11.  $u = f(x, y), \quad \text{where } x = x(r, s, t), \quad y = y(r, s, t)$
- 12.  $w = f(x, y, z), \quad \text{where } x = x(t, u), \quad y = y(t, u), \quad z = z(t, u)$
- 13.  $v = f(p, q, r),$   
where  $p = p(x, y, z), \quad q = q(x, y, z), \quad r = r(x, y, z)$

14.  $u = f(s, t), \quad \text{where } s = s(w, x, y, z), \quad t = t(w, x, y, z)$

15-19 ■ Use the Chain Rule to find the indicated partial derivatives.

15.  $w = x^2 + y^2 + z^2, \quad x = st, \quad y = s \cos t, \quad z = s \sin t;$   
 $\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t}$  when  $s = 1, t = 0$

16.  $u = xy + yz + zx, \quad x = st, \quad y = e^{st}, \quad z = t^2;$   
 $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}$  when  $s = 0, t = 1$

17.  $z = y^2 \tan x, \quad x = t^2 uv, \quad y = u + tv^2;$   
 $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$  when  $t = 2, u = 1, v = 0$

18.  $z = \frac{x}{y}, \quad x = re^{st}, \quad y = rse^t;$   
 $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$  when  $r = 1, s = 2, t = 0$

19.  $u = \frac{x + y}{y + z}, \quad x = p + r + t, \quad y = p - r + t,$   
 $z = p + r - t;$   
 $\frac{\partial u}{\partial p}, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t}$

20-22 ■ Use Equation 6 to find  $dy/dx$ .

20.  $y^5 + x^2y^3 = 1 + ye^{x^2}$

21.  $\cos(x - y) = xe^y$

22.  $\sin x + \cos y = \sin x \cos y$

23–26 ■ Use Equations 7 to find  $\partial z/\partial x$  and  $\partial z/\partial y$ .

23.  $xy^2 + yz^2 + zx^2 = 3$       24.  $xyz = \cos(x + y + z)$

25.  $xe^y + yz + ze^x = 0$       26.  $\ln(x + yz) = 1 + xy^2z^3$

27. The temperature at a point  $(x, y)$  is  $T(x, y)$ , measured in degrees Celsius. A bug crawls so that its position after  $t$  seconds is given by  $x = \sqrt{1+t}$ ,  $y = 2 + \frac{1}{3}t$ , where  $x$  and  $y$  are measured in centimeters. The temperature function satisfies  $T_x(2, 3) = 4$  and  $T_y(2, 3) = 3$ . How fast is the temperature rising on the bug's path after 3 seconds?

28. Wheat production in a given year,  $W$ , depends on the average temperature  $T$  and the annual rainfall  $R$ . Scientists estimate that the average temperature is rising at a rate of  $0.15^\circ\text{C}/\text{year}$  and rainfall is decreasing at a rate of  $0.1\text{ cm}/\text{year}$ . They also estimate that, at current production levels,  $\partial W/\partial T = -2$  and  $\partial W/\partial R = 8$ .

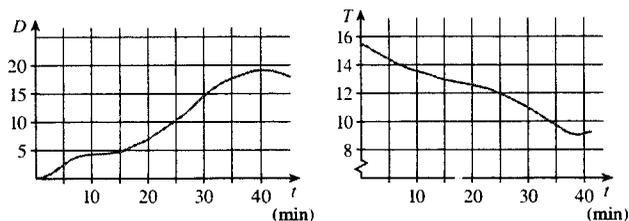
(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production,  $dW/dt$ .

29. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where  $C$  is the speed of sound (in meters per second),  $T$  is the temperature (in degrees Celsius), and  $D$  is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?



30. The radius of a right circular cone is increasing at a rate of  $1.8\text{ in}/\text{s}$  while its height is decreasing at a rate of  $2.5\text{ in}/\text{s}$ . At what rate is the volume of the cone changing when the radius is  $120\text{ in.}$  and the height is  $140\text{ in.}$ ?

31. The length  $\ell$ , width  $w$ , and height  $h$  of a box change with time. At a certain instant the dimensions are  $\ell = 1\text{ m}$  and  $w = h = 2\text{ m}$ , and  $\ell$  and  $w$  are increasing at a rate of  $2\text{ m}/\text{s}$

while  $h$  is decreasing at a rate of  $3\text{ m}/\text{s}$ . At that instant find the rates at which the following quantities are changing.

- (a) The volume      (b) The surface area  
(c) The length of a diagonal

32. The voltage  $V$  in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance  $R$  is slowly increasing as the resistor heats up. Use Ohm's Law,  $V = IR$ , to find how the current  $I$  is changing at the moment when  $R = 400\ \Omega$ ,  $I = 0.08\text{ A}$ ,  $dV/dt = -0.01\text{ V}/\text{s}$ , and  $dR/dt = 0.03\ \Omega/\text{s}$ .

33. The pressure of 1 mole of an ideal gas is increasing at a rate of  $0.05\text{ kPa}/\text{s}$  and the temperature is increasing at a rate of  $0.15\text{ K}/\text{s}$ . Use the equation in Example 2 to find the rate of change of the volume when the pressure is  $20\text{ kPa}$  and the temperature is  $320\text{ K}$ .

34. Car A is traveling north on Highway 16 and Car B is traveling west on Highway 83. Each car is approaching the intersection of these highways. At a certain moment, car A is  $0.3\text{ km}$  from the intersection and traveling at  $90\text{ km}/\text{h}$  while car B is  $0.4\text{ km}$  from the intersection and traveling at  $80\text{ km}/\text{h}$ . How fast is the distance between the cars changing at that moment?

35–38 ■ Assume that all the given functions are differentiable.

35. If  $z = f(x, y)$ , where  $x = r \cos \theta$  and  $y = r \sin \theta$ , (a) find  $\partial z/\partial r$  and  $\partial z/\partial \theta$  and (b) show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$$

36. If  $u = f(x, y)$ , where  $x = e^s \cos t$  and  $y = e^s \sin t$ , show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s} \left[ \left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 \right]$$

37. If  $z = f(x - y)$ , show that  $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ .

38. If  $z = f(x, y)$ , where  $x = s + t$  and  $y = s - t$ , show that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\partial z}{\partial s} \frac{\partial z}{\partial t}$$

39–44 ■ Assume that all the given functions have continuous second-order partial derivatives.

39. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

[Hint: Let  $u = x + at$ ,  $v = x - at$ .]

40. If  $u = f(x, y)$ , where  $x = e^s \cos t$  and  $y = e^s \sin t$ , show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s} \left[ \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} \right]$$

41. If  $z = f(x, y)$ , where  $x = r^2 + s^2$ ,  $y = 2rs$ , find  $\partial^2 z / \partial r \partial s$ . (Compare with Example 7.)

42. If  $z = f(x, y)$ , where  $x = r \cos \theta$ ,  $y = r \sin \theta$ , find (a)  $\partial z / \partial r$ , (b)  $\partial z / \partial \theta$ , and (c)  $\partial^2 z / \partial r \partial \theta$ .

43. If  $z = f(x, y)$ , where  $x = r \cos \theta$ ,  $y = r \sin \theta$ , show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r}$$

44. Suppose  $z = f(x, y)$ , where  $x = g(s, t)$  and  $y = h(s, t)$ . (a) Show that

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial^2 z}{\partial x^2} \left( \frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 z}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2} \left( \frac{\partial y}{\partial t} \right)^2 \\ &\quad + \frac{\partial z}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y} \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

- (b) Find a similar formula for  $\partial^2 z / \partial s \partial t$ .

45. Suppose that the equation  $F(x, y, z) = 0$  implicitly defines each of the three variables  $x$ ,  $y$ , and  $z$  as functions of the other two:  $z = f(x, y)$ ,  $y = g(x, z)$ ,  $x = h(y, z)$ . If  $F$  is differentiable and  $F_x, F_y$ , and  $F_z$  are all nonzero, show that

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1$$

**11.6**

**Directional Derivatives and the Gradient Vector**

The weather map in Figure 1 shows a contour map of the temperature function  $T(x, y)$  for the states of California and Nevada at 3:00 P.M. on October 10, 1997. The level curves, or isothermals, join locations with the same temperature. The partial derivative  $T_x$  at a location such as Reno is the rate of change of temperature with respect to distance if we travel east from Reno;  $T_y$  is the rate of change of temperature if we travel north. But what if we want to know the rate of change of temperature when we travel southeast (toward Las Vegas), or in some other direction? In this section we introduce a type of derivative, called a *directional derivative*, that enables us to find the rate of change of a function of two or more variables in any direction.

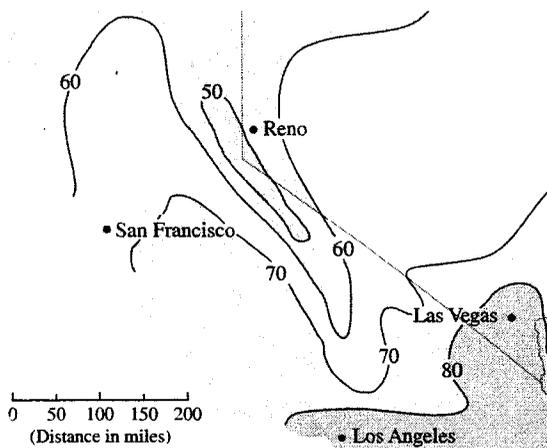


FIGURE 1

**Directional Derivatives**

Recall that if  $z = f(x, y)$ , then the partial derivatives  $f_x$  and  $f_y$  are defined as

$$\begin{aligned} f_x(x_0, y_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\ f_y(x_0, y_0) &= \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \end{aligned}$$