Computer algebra systems have commands that plot sample gradient vectors. Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure 13 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f. As expected, the gradient vectors point "uphill" and are perpendicular to the level curves.

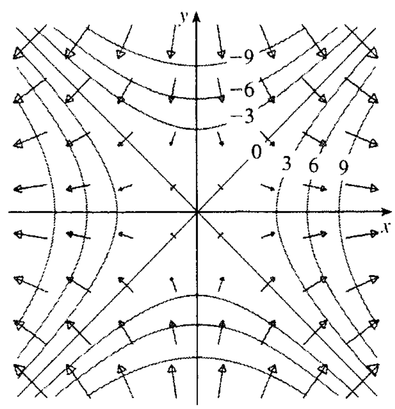
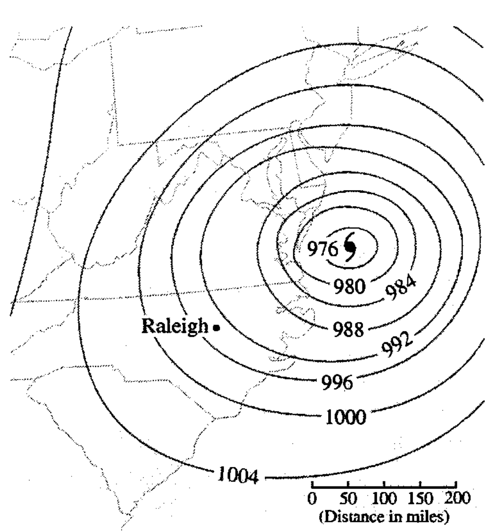


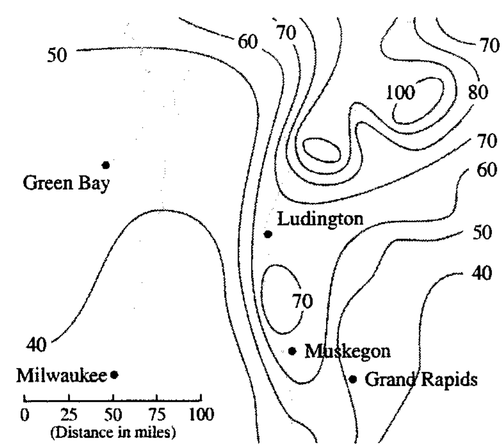
FIGURE 13

11.6 Exercises

1. A contour map of barometric pressure (in millibars) is shown for 7:00 A.M. on September 12, 1960, when Hurricane Donna was raging. Estimate the value of the directional derivative of the pressure function at Raleigh, North Carolina, in the direction of the eye of the hurricane. What are the units of the directional derivative?



2. The contour map shows the average annual snowfall (in inches) near Lake Michigan. Estimate the value of the directional derivative of this snowfall function at Muskegon, Michigan, in the direction of Ludington. What are the units?



3. A table of values for the wind chill index $I = f(T, v)$ is given in Exercise 3 on page 776. Use the table to estimate the value of $D_{\mathbf{u}} f(16, 30)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

4–6 ■ Find the directional derivative of f at the given point in the direction indicated by the angle θ.

4. $f(x, y) = \sin(x + 2y)$, $(4, -2)$, $\theta = 3\pi/4$

5. $f(x, y) = \sqrt{5x - 4y}$, $(4, 1)$, $\theta = -\pi/6$

6. $f(x, y) = xe^{-2y}$, $(5, 0)$, $\theta = \pi/2$

7–10 ■
(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = 5xy^2 - 4x^3y$, $P(1, 2)$, $\mathbf{u} = \langle \frac{5}{13}, \frac{12}{13} \rangle$

8. $f(x, y) = y \ln x$, $P(1, -3)$, $\mathbf{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle$

9. $f(x, y, z) = xy^2z^3$, $P(1, -2, 1)$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

10. $f(x, y, z) = xy + yz^2 + xz^3$, $P(2, 0, 3)$, $\mathbf{u} = \langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle$

11–15 ■ Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$

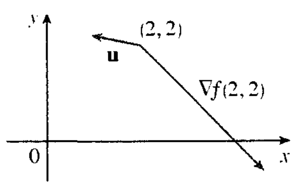
12. $g(r, \theta) = e^{-r} \sin \theta$, $(0, \pi/3)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

13. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(1, 2, -2)$, $\mathbf{v} = \langle -6, 6, -3 \rangle$

14. $f(x, y, z) = x/(y + z)$, $(4, 1, 1)$, $\mathbf{v} = \langle 1, 2, 3 \rangle$

15. $g(x, y, z) = x \tan^{-1}(y/z)$, $(1, 2, -2)$, $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

16. Use the figure to estimate $D_{\mathbf{u}} f(2, 2)$.



17. Find the directional derivative of $f(x, y) = \sqrt{xy}$ at $P(2, 8)$ in the direction of $Q(5, 4)$.

18. Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at $P(2, 1, 3)$ in the direction of the origin.

19–22 ■ Find the maximum rate of change of f at the given point and the direction in which it occurs.

19. $f(x, y) = \sin(xy)$, $(1, 0)$

20. $f(x, y) = \ln(x^2 + y^2)$, $(1, 2)$

21. $f(x, y, z) = x + y/z$, $(4, 3, -1)$

22. $f(x, y, z) = x^2y^3z^4$, $(1, 1, 1)$

23. (a) Show that a differentiable function f decreases most rapidly at $\mathbf{x}$ in the direction opposite to the gradient vector, that is, in the direction of $-\nabla f(\mathbf{x})$.
(b) Use the result of part (a) to find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases fastest at the point $(2, -3)$.

24. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ at the point $(1, 0)$ has the value 1.

25. Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\mathbf{i} + \mathbf{j}$.

26. Near a buoy, the depth of a lake at the point with coordinates (x, y) is $z = 200 + 0.02x^2 - 0.001y^3$, where x, y, and z are measured in meters. A fisherman in a small boat starts at the point $(80, 60)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

27. The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is 120°.
(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin.

28. The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where T is measured in °C and x, y, z in meters.
(a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.
(b) In which direction does the temperature increase fastest at P?
(c) Find the maximum rate of increase at P.

29. Suppose that over a certain region of space the electrical potential V is given by

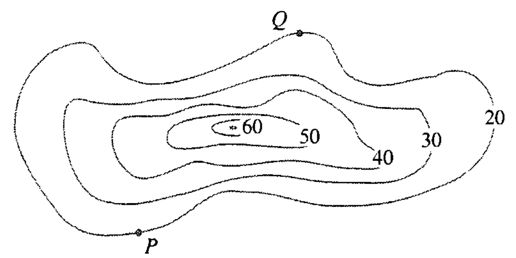
$$V(x, y, z) = 5x^2 - 3xy + xyz$$

(a) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.
(b) In which direction does V change most rapidly at P?
(c) What is the maximum rate of change at P?

30. Suppose that you are climbing a hill whose shape is given by the equation $z = 1000 - 0.01x^2 - 0.02y^2$ and you are standing at a point with coordinates $(60, 100, 764)$.
(a) In which direction should you proceed initially in order to reach the top of the hill fastest?
(b) If you climb in that direction, at what angle above the horizontal will you be climbing initially?

31. Let f be a function of two variables that has continuous partial derivatives and consider the points $A(1, 3)$, $B(3, 3)$, $C(1, 7)$, and $D(6, 15)$. The directional derivative of f at A in the direction of the vector $\overrightarrow{AB}$ is 3 and the directional derivative at A in the direction of $\overrightarrow{AC}$ is 26. Find the directional derivative of f at A in the direction of the vector $\overrightarrow{AD}$.

32. For the given contour map draw the curves of steepest ascent starting at P and at Q.



33. Show that the operation of taking the gradient of a function has the given property. Assume that u and v are differentiable functions of x and y and a, b are constants.

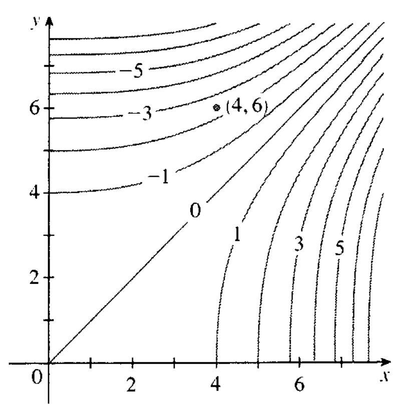
(a) $\nabla(au + bv) = a\,\nabla u + b\,\nabla v$

(b) $\nabla(uv) = u\,\nabla v + v\,\nabla u$

(c) $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = nu^{n-1}\,\nabla u$

34. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.



35–38 ■ Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

35. $x^2 + 2y^2 + 3z^2 = 21, \quad (4, -1, 1)$

36. $x = y^2 + z^2 - 2, \quad (-1, 1, 0)$

37. $z + 1 = xe^y \cos z, \quad (1, 0, 0)$

38. $xe^{yz} = 1, \quad (1, 0, 5)$

39–40 ■ Use a computer to graph the surface, the tangent plane, and the normal line on the same screen. Choose the domain carefully so that you avoid extraneous vertical planes. Choose the viewpoint so that you get a good view of all three objects.

39. $xy + yz + zx = 3, \quad (1, 1, 1)$

40. $xyz = 6, \quad (1, 2, 3)$

41. If $f(x, y) = x^2 + 4y^2$, find the gradient vector $\nabla f(2, 1)$ and use it to find the tangent line to the level curve $f(x, y) = 8$ at the point $(2, 1)$. Sketch the level curve, the tangent line, and the gradient vector.

42. If $g(x, y) = x - y^2$, find the gradient vector $\nabla g(3, -1)$ and use it to find the tangent line to the level curve $g(x, y) = 2$ at the point $(3, -1)$. Sketch the level curve, the tangent line, and the gradient vector.

43. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

44. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$.

45. Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3, -1, 0)$ and $(5, 3, 6)$.

46. Show that the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$. (This means that they have a common tangent plane at the point.)

47. Show that the sum of the x-, y-, and z-intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

48. Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

49. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

50. (a) The plane $y + z = 3$ intersects the cylinder $x^2 + y^2 = 5$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 1)$.

(b) Graph the cylinder, the plane, and the tangent line on the same screen.

51. (a) Two surfaces are called **orthogonal** at a point of intersection if their normal lines are perpendicular at that point. Show that surfaces with equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ are orthogonal at a point P where $\nabla F \neq \mathbf{0}$ and $\nabla G \neq \mathbf{0}$ if and only if

$$F_xG_x + F_yG_y + F_zG_z = 0$$

at P.

(b) Use part (a) to show that the surfaces $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = r^2$ are orthogonal at every point of intersection. Can you see why this is true without using calculus?

52. (a) Show that the function $f(x, y) = \sqrt[3]{xy}$ is continuous and the partial derivatives f_x and f_y exist at the origin but the directional derivatives in all other directions do not exist.