

This is a decreasing function of  $y$ , so its maximum value is  $f(3, 0) = 9$  and its minimum value is  $f(3, 2) = 1$ . On  $L_3$  we have  $y = 2$  and

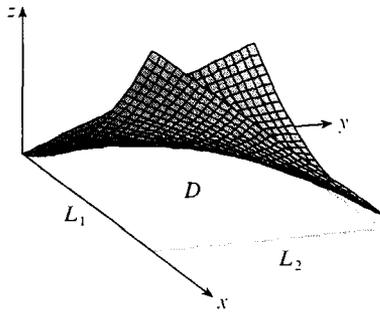
$$f(x, 2) = x^2 - 4x + 4 \quad 0 \leq x \leq 3$$

By the methods of Chapter 4, or simply by observing that  $f(x, 2) = (x - 2)^2$ , we see that the minimum value of this function is  $f(2, 2) = 0$  and the maximum value is  $f(0, 2) = 4$ . Finally, on  $L_4$  we have  $x = 0$  and

$$f(0, y) = 2y \quad 0 \leq y \leq 2$$

with maximum value  $f(0, 2) = 4$  and minimum value  $f(0, 0) = 0$ . Thus, on the boundary, the minimum value of  $f$  is 0 and the maximum is 9.

In step 3 we compare these values with the value  $f(1, 1) = 1$  at the critical point and conclude that the absolute maximum value of  $f$  on  $D$  is  $f(3, 0) = 9$  and the absolute minimum value is  $f(0, 0) = f(2, 2) = 0$ . Figure 13 shows the graph of  $f$ .



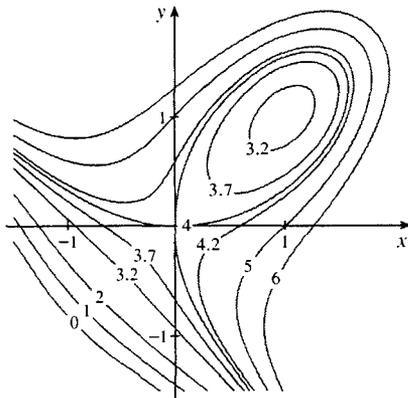
**FIGURE 13**  
 $f(x, y) = x^2 - 2xy + 2y^2$

**11.7 Exercises**

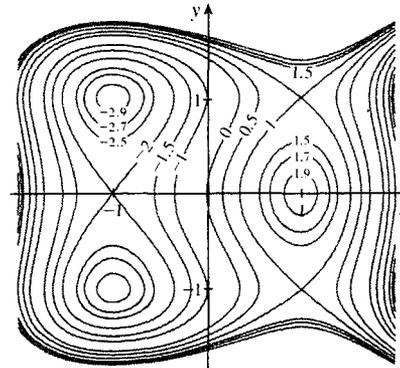
1. Suppose  $(1, 1)$  is a critical point of a function  $f$  with continuous second derivatives. In each case, what can you say about  $f$ ?
  - (a)  $f_{xx}(1, 1) = 4, \quad f_{yy}(1, 1) = 1, \quad f_{xy}(1, 1) = 2$
  - (b)  $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 3, \quad f_{yy}(1, 1) = 2$
2. Suppose  $(0, 2)$  is a critical point of a function  $g$  with continuous second derivatives. In each case, what can you say about  $g$ ?
  - (a)  $g_{xx}(0, 2) = -1, \quad g_{yy}(0, 2) = 6, \quad g_{xy}(0, 2) = 1$
  - (b)  $g_{xx}(0, 2) = -1, \quad g_{xy}(0, 2) = 2, \quad g_{yy}(0, 2) = -8$
  - (c)  $g_{xx}(0, 2) = 4, \quad g_{xy}(0, 2) = 6, \quad g_{yy}(0, 2) = 9$

3–4 ■ Use the level curves in the figure to predict the location of the critical points of  $f$  and whether  $f$  has a saddle point or a local maximum or minimum at each of those points. Explain your reasoning. Then use the Second Derivatives Test to confirm your predictions.

3.  $f(x, y) = 4 + x^3 + y^3 - 3xy$



4.  $f(x, y) = 3x - x^3 - 2y^2 + y^4$



5–14 ■ Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5.  $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$
6.  $f(x, y) = x^3y + 12x^2 - 8y$
7.  $f(x, y) = x^2 + y^2 + x^2y + 4$
8.  $f(x, y) = e^{4y-x^2-y^2}$
9.  $f(x, y) = xy - 2x - y$
10.  $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$
11.  $f(x, y) = e^x \cos y$
12.  $f(x, y) = x^2 + y^2 + \frac{1}{x^2y^2}$
13.  $f(x, y) = x \sin y$
14.  $f(x, y) = (2x - x^2)(2y - y^2)$

 **15–18** ■ Use a graph and/or level curves to estimate the local maximum and minimum values and saddle point(s) of the function. Then use calculus to find these values precisely.

15.  $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

16.  $f(x, y) = xye^{-x^2-y^2}$

17.  $f(x, y) = \sin x + \sin y + \sin(x + y)$ ,  
 $0 \leq x \leq 2\pi, 0 \leq y \leq 2\pi$

18.  $f(x, y) = \sin x + \sin y + \cos(x + y)$ ,  
 $0 \leq x \leq \pi/4, 0 \leq y \leq \pi/4$

 **19–22** ■ Use a graphing device as in Example 4 (or Newton's method or a rootfinder) to find the critical points of  $f$  correct to three decimal places. Then classify the critical points and find the highest or lowest points on the graph.

19.  $f(x, y) = x^4 - 5x^2 + y^2 + 3x + 2$

20.  $f(x, y) = 5 - 10xy - 4x^2 + 3y - y^4$

21.  $f(x, y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4$

22.  $f(x, y) = e^x + y^4 - x^3 + 4 \cos y$

**23–28** ■ Find the absolute maximum and minimum values of  $f$  on the set  $D$ .

23.  $f(x, y) = 1 + 4x - 5y$ .  $D$  is the closed triangular region with vertices  $(0, 0)$ ,  $(2, 0)$ , and  $(0, 3)$

24.  $f(x, y) = 3 + xy - x - 2y$ .  $D$  is the closed triangular region with vertices  $(1, 0)$ ,  $(5, 0)$ , and  $(1, 4)$

25.  $f(x, y) = x^2 + y^2 + x^2y + 4$ ,  
 $D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$

26.  $f(x, y) = 4x + 6y - x^2 - y^2$ ,  
 $D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 5\}$

27.  $f(x, y) = 1 + xy - x - y$ .  $D$  is the region bounded by the parabola  $y = x^2$  and the line  $y = 4$

28.  $f(x, y) = xy^2$ ,  $D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

 **29.** For functions of one variable it is impossible for a continuous function to have two local maxima and no local minimum. But for functions of two variables such functions exist. Show that the function

$$f(x, y) = -(x^2 - 1)^2 - (x^2y - x - 1)^2$$

has only two critical points, but has local maxima at both of them. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

 **30.** If a function of one variable is continuous on an interval and has only one critical number, then a local maximum has to be an absolute maximum. But this is not true for functions of two variables. Show that the function

$$f(x, y) = 3xe^y - x^3 - e^{3y}$$

has exactly one critical point, and that  $f$  has a local maximum there that is not an absolute maximum. Then use a computer to produce a graph with a carefully chosen domain and viewpoint to see how this is possible.

- 31.** Find the shortest distance from the point  $(2, 1, -1)$  to the plane  $x + y - z = 1$ .
- 32.** Find the point on the plane  $x - y + z = 4$  that is closest to the point  $(1, 2, 3)$ .
- 33.** Find the points on the surface  $z^2 = xy + 1$  that are closest to the origin.
- 34.** Find the points on the surface  $x^2y^2z = 1$  that are closest to the origin.
- 35.** Find three positive numbers whose sum is 100 and whose product is a maximum.
- 36.** Find three positive numbers  $x$ ,  $y$ , and  $z$  whose sum is 100 such that  $x^ay^bz^c$  is a maximum.
- 37.** Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid

$$9x^2 + 36y^2 + 4z^2 = 36$$

- 38.** Solve the problem in Exercise 37 for a general ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- 39.** Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane  $x + 2y + 3z = 6$ .
- 40.** Find the dimensions of the rectangular box with largest volume if the total surface area is given as  $64 \text{ cm}^2$ .
- 41.** Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant  $c$ .
- 42.** The base of an aquarium with given volume  $V$  is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.
- 43.** A cardboard box without a lid is to have a volume of  $32,000 \text{ cm}^3$ . Find the dimensions that minimize the amount of cardboard used.
- 44.** Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the proportion of individuals in a population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where  $p$ ,  $q$ , and  $r$  are the proportions of A, B, and O in the population. Use the fact that  $p + q + r = 1$  to show that  $P$  is at most  $\frac{2}{3}$ .

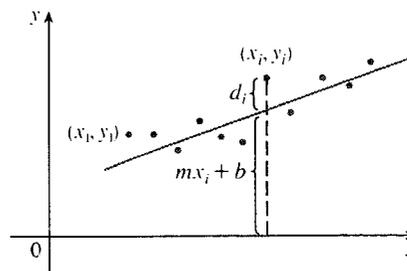
- 45.** Suppose that a scientist has reason to believe that two quantities  $x$  and  $y$  are related linearly, that is,  $y = mx + b$ , at

least approximately, for some values of  $m$  and  $b$ . The scientist performs an experiment and collects data in the form of points  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants  $m$  and  $b$  so that the line  $y = mx + b$  "fits" the points as well as possible. (See the figure.) Let  $d_i = y_i - (mx_i + b)$  be the vertical deviation of the point  $(x_i, y_i)$  from the line. The **method of least squares** determines  $m$  and  $b$  so as to minimize  $\sum_{i=1}^n d_i^2$ , the sum of the squares of these deviations. Show that, according to this method, the line of best fit is obtained when

$$m \sum_{i=1}^n x_i + bn = \sum_{i=1}^n y_i$$

$$m \sum_{i=1}^n x_i^2 + b \sum_{i=1}^n x_i = \sum_{i=1}^n x_i y_i$$

Thus, the line is found by solving these two equations in the two unknowns  $m$  and  $b$ . (See Section 1.2 for a further discussion and applications of the method of least squares.)



46. Find an equation of the plane that passes through the point  $(1, 2, 3)$  and cuts off the smallest volume in the first octant.

## Applied Project

### Designing a Dumpster

For this project we locate a trash dumpster in order to study its shape and construction. We then attempt to determine the dimensions of a container of similar design that minimize construction cost.

1. First locate a trash dumpster in your area. Carefully study and describe all details of its construction, and determine its volume. Include a sketch of the container.
2. While maintaining the general shape and method of construction, determine the dimensions such a container of the same volume should have in order to minimize the cost of construction. Use the following assumptions in your analysis:
  - The sides, back, and front are to be made from 12-gauge (0.1046 inch thick) steel sheets, which cost \$0.70 per square foot (including any required cuts or bends).
  - The base is to be made from a 10-gauge (0.1345 inch thick) steel sheet, which costs \$0.90 per square foot.
  - Lids cost approximately \$50.00 each, regardless of dimensions.
  - Welding costs approximately \$0.18 per foot for material and labor combined.

Give justification of any further assumptions or simplifications made of the details of construction.

3. Describe how any of your assumptions or simplifications may affect the final result.
4. If you were hired as a consultant on this investigation, what would your conclusions be? Would you recommend altering the design of the dumpster? If so, describe the savings that would result.