

▲ The cylinder $x^2 + y^2 = 1$ intersects the plane $x - y + z = 1$ in an ellipse (Figure 6). Example 5 asks for the maximum value of f when (x, y, z) is restricted to lie on the ellipse.

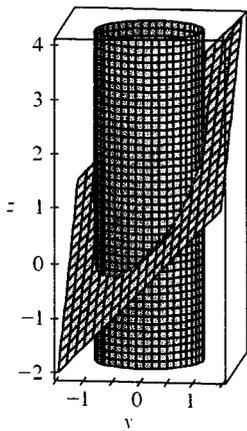


FIGURE 6

EXAMPLE 5 Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

SOLUTION We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$. The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

$$\begin{aligned} \boxed{17} \quad & 1 = \lambda + 2x\mu \\ \boxed{18} \quad & 2 = -\lambda + 2y\mu \\ \boxed{19} \quad & 3 = \lambda \\ \boxed{20} \quad & x - y + z = 1 \\ \boxed{21} \quad & x^2 + y^2 = 1 \end{aligned}$$

Putting $\lambda = 3$ [from (19)] in (17), we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, (18) gives $y = 5/(2\mu)$. Substitution in (21) then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$. Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from (20), $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. The corresponding values of f are

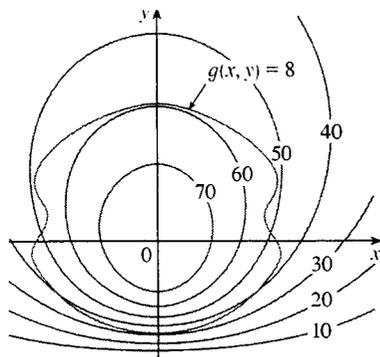
$$\mp \frac{2}{\sqrt{29}} + 2\left(\pm \frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore, the maximum value of f on the given curve is $3 + \sqrt{29}$.

11.8

Exercises

1. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



2. (a) Use a graphing calculator or computer to graph the circle $x^2 + y^2 = 1$. On the same screen, graph several curves of the form $x^2 + y = c$ until you find two that just touch the circle. What is the significance of the values of c for these two curves?
 (b) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. Compare your answers with those in part (a).

3-17 ■ Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

3. $f(x, y) = x^2 - y^2$; $x^2 + y^2 = 1$
4. $f(x, y) = 4x + 6y$; $x^2 + y^2 = 13$
5. $f(x, y) = x^2y$; $x^2 + 2y^2 = 6$
6. $f(x, y) = x^2 + y^2$; $x^4 + y^4 = 1$

- 7. $f(x, y, z) = 2x + 6y + 10z$; $x^2 + y^2 + z^2 = 35$
- 8. $f(x, y, z) = 8x - 4z$; $x^2 + 10y^2 + z^2 = 5$
- 9. $f(x, y, z) = xyz$; $x^2 + 2y^2 + 3z^2 = 6$
- 10. $f(x, y, z) = x^2y^2z^2$; $x^2 + y^2 + z^2 = 1$
- 11. $f(x, y, z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$
- 12. $f(x, y, z) = x^4 + y^4 + z^4$; $x^2 + y^2 + z^2 = 1$
- 13. $f(x, y, z, t) = x + y + z + t$; $x^2 + y^2 + z^2 + t^2 = 1$
- 14. $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$;
 $x_1^2 + x_2^2 + \dots + x_n^2 = 1$
- 15. $f(x, y, z) = x + 2y$; $x + y + z = 1$, $y^2 + z^2 = 4$
- 16. $f(x, y, z) = 3x - y - 3z$;
 $x + y - z = 0$, $x^2 + 2z^2 = 1$
- 17. $f(x, y, z) = yz + xy$; $xy = 1$, $y^2 + z^2 = 1$

18–19 ■ Find the extreme values of f on the region described by the inequality.

- 18. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$, $x^2 + y^2 \leq 16$
- 19. $f(x, y) = e^{-xy}$, $x^2 + 4y^2 \leq 1$

- CAS** 20. (a) If your computer algebra system plots implicitly defined curves, use it to estimate the minimum and maximum values of $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$ by graphical methods. (b) Solve the problem in part (a) with the aid of Lagrange multipliers. Use your CAS to solve the equations. Compare your answers with those in part (a).

21. The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment. In Sections 11.1 and 11.3 we discussed how the Cobb-Douglas model $P = bL^\alpha K^{1-\alpha}$ follows from certain economic assumptions, where b and α are positive constants and $\alpha < 1$. If the cost of a unit of labor is m and the cost of a unit of capital is n , and the company can spend only p dollars as its total budget, then maximizing the production P is subject to the constraint $mL + nK = p$. Show that the maximum production occurs when

$$L = \frac{\alpha p}{m} \quad \text{and} \quad K = \frac{(1 - \alpha)p}{n}$$

22. Referring to Exercise 21, we now suppose that the production is fixed at $bL^\alpha K^{1-\alpha} = Q$, where Q is a constant. What values of L and K minimize the cost function $C(L, K) = mL + nK$?

23. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

24. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral. [Hint: Use Heron's formula for the area: $A = \sqrt{s(s-x)(s-y)(s-z)}$, where $s = p/2$ and x, y, z are the lengths of the sides.]

25–35 ■ Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 11.7.

- 25. Exercise 31
- 26. Exercise 32
- 27. Exercise 33
- 28. Exercise 34
- 29. Exercise 35
- 30. Exercise 36
- 31. Exercise 37
- 32. Exercise 38
- 33. Exercise 39
- 34. Exercise 40
- 35. Exercise 41

36. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm .

37. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

38. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.

- (a) Graph the cone, the plane, and the ellipse.
- (b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

CAS 39–40 ■ Find the maximum and minimum values of f subject to the given constraints. Use a computer algebra system to solve the system of equations that arises in using Lagrange multipliers. (If your CAS finds only one solution, you may need to use additional commands.)

39. $f(x, y, z) = ye^{x-z}$; $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$

40. $f(x, y, z) = x + y + z$; $x^2 - y^2 = z$, $x^2 + z^2 = 4$

41. (a) Find the maximum value of

$$f(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that x_1, x_2, \dots, x_n are positive numbers and $x_1 + x_2 + \dots + x_n = c$, where c is a constant.

(b) Deduce from part (a) that if x_1, x_2, \dots, x_n are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal to each other?

42. (a) Maximize $\sum_{i=1}^n x_i y_i$ subject to the constraints $\sum x_i^2 = 1$
and $\sum y_i^2 = 1$.
(b) Put

$$x_i = \frac{a_i}{\sqrt{\sum a_i^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\sum b_i^2}}$$

to show that

$$\sum a_i b_i \leq \sqrt{\sum a_i^2} \sqrt{\sum b_i^2}$$

for any numbers $a_1, \dots, a_n, b_1, \dots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

Applied Project

Rocket Science

Many rockets, such as the *Pegasus XL* currently used to launch satellites and the *Saturn V* that first put men on the Moon, are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about Earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages to be designed in such a way as to minimize the total mass of the rocket while enabling it to reach a desired velocity.

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta V = -c \ln \left(1 - \frac{(1-S)M_r}{P + M_r} \right)$$

where M_r is the mass of the rocket engine including initial fuel, P is the mass of the payload, S is a *structural factor* determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload), and c is the (constant) speed of exhaust relative to the rocket.

Now consider a rocket with three stages and a payload of mass A . We will consider outside forces negligible and assume that c and S remain constant for each stage. If M_i is the mass of the i th stage, we can initially consider the rocket engine to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

1. Show that the velocity attained after all three stages have been jettisoned is given by

$$v_f = c \left[\ln \left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} \right) + \ln \left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A} \right) + \ln \left(\frac{M_3 + A}{SM_3 + A} \right) \right]$$

2. We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the desired velocity v_f from Problem 1 is attained. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions. To simplify, we define variables N_i so that the constraint equation may be expressed as $v_f = c(\ln N_1 + \ln N_2 + \ln N_3)$. Since M is now difficult to express in terms of the N_i 's, we wish to use a simpler function that will be minimized at the same place. Show that

$$\frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} = \frac{(1-S)N_1}{1 - SN_1}$$

$$\frac{M_2 + M_3 + A}{M_3 + A} = \frac{(1-S)N_2}{1 - SN_2}$$

$$\frac{M_3 + A}{A} = \frac{(1-S)N_3}{1 - SN_3}$$