Solutions to B1-4 of Problem Set 5

B1

We want to maximize the function $u(x, y) = x^\alpha y^\beta$ subject to the constraints $x \geq 0$, $y \geq 0$ and $px + qy \leq b$. These are inequalities, and so we have to solve this problem in 2 steps.

i) Interior: We find the critical points in the interior by setting the gradient equal to zero and simply choosing the solutions which fit the constraints.

$$\begin{aligned} u_x &= \alpha x^{\alpha-1} y^\beta = 0 \\ u_y &= \beta x^\alpha y^{\beta-1} = 0 \end{aligned}$$

If critical points exist then they are all the points on the two axes. However note that if α and/or β are less than 1 then there are no critical points, since the respective partial derivative blows up as x and/or y approach 0. This is not crucial in this particular problem because we're looking for the maximum value only and 0 happens to be a minimum value here, however in other problems this might be more important so take note of the subtlety.

The value of $u(x, y)$ along both axes is zero.

ii) Boundary: We use Lagrange's equation to find the critical points on the boundary. Since the critical points from part i) happen to be on 2 of the 3 boundaries, those are taken care of and we can focus on the third boundary: $px + qy = b$. Define $g(x, y) = px + qy - b$. Notice that I have changed the inequality to an equality, because I'm looking only at the line, I've already considered the interior. Applying Lagrange's equation, we get

$$u_x = \lambda g_x \;\Rightarrow\; \alpha x^{\alpha-1} y^\beta = \lambda p$$

$$u_y = \lambda g_y \ \Rightarrow \ \beta x^\alpha y^{\beta-1} = \lambda q.$$

It is very tempting to divide these two equations to eliminate λ, however before we can do that we have to make sure we're not dividing by zero. Since $p,\ q \geq 0$ and $\lambda \neq 0$, the right hand side of both equations is non-zero and we can safely divide them. This yields

$$\frac{\alpha y}{\beta x} = \frac{p}{q} \Rightarrow y = \frac{\beta p}{\alpha q} x$$

Since we have eliminated λ we can plug this relation into our constraint and solve for x:

$$px + q\frac{\beta p}{\alpha q}x = \frac{\alpha+\beta}{\alpha}px = b \Rightarrow x_{crit} = \frac{\alpha b}{(\alpha+\beta)p} \Rightarrow y_{crit} = \frac{\beta b}{(\alpha+\beta)q}$$

and in turn

$$u(x_{crit},\ y_{crit}) = \left(\frac{\alpha b}{(\alpha+\beta)p}\right)^\alpha \left(\frac{\beta b}{(\alpha+\beta)q}\right)^\beta = \left(\frac{\alpha^\alpha \beta^\beta}{(\alpha+\beta)^{(\alpha+\beta)} p^\alpha q^\beta}\right) b^{(\alpha+\beta)}$$

This is the only critical point and what is left is to determing whether it is a minimum or a maximum. We know that all the constants are positive and so this must be a positive quantity. We also know that u is zero at the axes, and since a positive number is greater than zero, this must be a maximum.

B2

We have to show that $\frac{du}{db} = \lambda$ where $u(b) = u(x_{crit}(b), y_{crit}(b))$. To evaluate this derivattive we use the chain rule:

$$\frac{du}{db} = \frac{\partial u}{\partial x}\frac{dx}{db} + \frac{\partial u}{\partial y}\frac{dy}{db}.$$

Since we are only interested in the points that satisfy Lagrange's equations from B1, we can use $\frac{\partial u}{\partial x} = \lambda p$ and $\frac{\partial u}{\partial y} = \lambda q$. We also evaluate the total derivatives:

$$\frac{dx}{db} = \frac{\alpha}{(\alpha+\beta)p}, \quad \frac{dy}{db} = \frac{\beta}{(\alpha+\beta)q}$$

Plugging in to our chain rule, we get

$$\begin{aligned}\frac{du}{db} &= \lambda p \frac{\alpha}{(\alpha+\beta)p} + \lambda q \frac{\beta}{(\alpha+\beta)q} \\ &= \frac{\lambda\alpha}{\alpha+\beta} + \frac{\lambda\beta}{\alpha+\beta} \\ &= \lambda\frac{\alpha+\beta}{\alpha+\beta} \\ &= \lambda\end{aligned}$$

Notice that we did not have to calculate λ explicitly here.

B3

For this question we treat u as a function not only of b but also of p, the price of the first good.

a

If the government levies an income tax, that means that $b \to b - \rho$, in other words $db = -\rho$ (and $dp = 0$). To calculate the change in utility, u, we use the formula

$$\begin{aligned}du &= \frac{\partial u}{\partial b}db + \frac{\partial u}{\partial p}dp \\ &= \left(\frac{\alpha}{p}\right)^{\alpha}\left(\frac{\beta}{q}\right)^{\beta}\frac{(\alpha+\beta)b^{\alpha+\beta-1}}{(\alpha+\beta)^{\alpha+\beta}}(-\rho) + 0 \\ &= -\rho\left(\frac{\alpha}{p}\right)^{\alpha}\left(\frac{\beta}{q}\right)^{\beta}\left(\frac{b}{\alpha+\beta}\right)^{\alpha+\beta-1}\end{aligned}$$

b

Similarly to part a, using $dp = +\tau$ and $db = 0$:

$$du = \frac{\partial u}{\partial b}db + \frac{\partial u}{\partial p}dp$$

$$
\begin{aligned}
&= \left(-\alpha \frac{\alpha^\alpha}{p^{\alpha+1}}\right)\left(\frac{\beta}{q}\right)^\beta \left(\frac{b}{\alpha+\beta}\right)^{\alpha+\beta}\tau + 0 \\
&= -\tau \left(\frac{\alpha}{p}\right)^{\alpha+1}\left(\frac{\beta}{q}\right)^\beta \left(\frac{b}{\alpha+\beta}\right)^{\alpha+\beta}
\end{aligned}
$$

c

Here we set $db = x_{crit}dp$, namely $\rho = \left(\frac{\alpha}{p}\right)\left(\frac{b}{\alpha+\beta}\right)\tau$. We want to know in which case u decreases by a smaller amount.

$$
-\left(\frac{\alpha}{p}\right)\left(\frac{b}{\alpha+\beta}\right)\tau\left(\frac{\alpha}{p}\right)^\alpha\left(\frac{\beta}{q}\right)^\beta\left(\frac{b}{\alpha+\beta}\right)^{\alpha+\beta-1} \quad ? \quad -\tau\left(\frac{\alpha}{p}\right)^{\alpha+1}\left(\frac{\beta}{q}\right)^\beta\left(\frac{b}{\alpha+\beta}\right)^{\alpha+\beta}
$$

The two sides are equal, and so du is the same for both. The consumer doesn't care which kind of tax the government levies. This makes sense, since if the amount that the government gets is the same in both cases, the amount that the consumer has to give must also be the same. Money doesn't grow on trees.

B4

We must minimize the cost $C(x,\ y) = \frac{1}{2}xd + ey$ subject to the constraint $xy = X$. This is a strict equality, and so we use Lagrange multipliers. First we define $g(x,y) = xy - X$ so that we obtain:

$$
\begin{aligned}
C_x = \lambda g_x \ &\Rightarrow\ \frac{d}{2} = \lambda y \ \Rightarrow\ y = \frac{d}{2\lambda} \\
C_y = \lambda g_y \ &\Rightarrow\ e = \lambda x \ \Rightarrow\ x = \frac{e}{\lambda}.
\end{aligned}
$$

Plugging these values into our constraint equation and solving for λ we get

$$
xy = \frac{e}{\lambda}\frac{d}{2\lambda} = \frac{de}{2\lambda^2} = X \ \Rightarrow\ \lambda = \pm\sqrt{\frac{de}{2X}}.
$$

Since we are only dealing with positive quantities here, namely $x,\ y,\ e,\ d > 0$, we only take the positive value. Thus $x = \sqrt{\frac{2eX}{d}}$ and $y = \sqrt{\frac{de}{2X}}$. Thus the minimum value of C occurs at $(\sqrt{2eX/d}, \sqrt{de/2X})$, and the minimum cost is

$$C(\sqrt{\frac{2eX}{d}}, \sqrt{\frac{de}{2X}}) = \sqrt{2deX}.$$