

Solution to C of Problem Set 7

The only relationship between each of the sections is that the number of cars flowing in each junction is the same as the number flowing out. Let's see what equations this gives us:

$$\begin{array}{rclcl}
 \text{Junction A:} & x_1 & -x_2 & & = & 100 \\
 \text{Junction B:} & & x_2 & -x_3 & & = & -50 \\
 \text{Junction C:} & & & x_3 & -x_4 & & = & 120 \\
 \text{Junction D:} & & & & x_4 & -x_5 & & = & -150 \\
 \text{Junction E:} & & & & & x_5 & -x_6 & = & 80 \\
 \text{Junction F:} & -x_1 & & & & & +x_6 & = & 100
 \end{array}$$

Notice that the first 5 equations add up to the negative of the sixth. This means that there are only 5 independent equations even though there are 6 junctions. This makes sense physically since there is one degree of freedom: There could be any number of cars just going around and around in circles, not entering or leaving the roundabout. Thus we can add or subtract any constant to each of the x_i 's and still have a solution.

Now to solve the equations we switch to matrix notation and put it in row-reduced echelon form:

$$\left[\begin{array}{cccccc|c}
 1 & -1 & 0 & 0 & 0 & 0 & 100 \\
 0 & 1 & -1 & 0 & 0 & 0 & -50 \\
 0 & 0 & 1 & -1 & 0 & 0 & 120 \\
 0 & 0 & 0 & 1 & -1 & 0 & -150 \\
 0 & 0 & 0 & 0 & 1 & -1 & 80
 \end{array} \right] \rightarrow \left[\begin{array}{cccccc|c}
 1 & 0 & 0 & 0 & 0 & -1 & 100 \\
 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 1 & 0 & 0 & -1 & 50 \\
 0 & 0 & 0 & 1 & 0 & -1 & -70 \\
 0 & 0 & 0 & 0 & 1 & -1 & 80
 \end{array} \right]$$

To get the row-reduced matrix, I added row 5 to row 4, rows 5 and 4 to row 3, rows 5, 4 and 3 to row 2, and rows 5, 4, 3 and 2 to row 1. Thus we see that

$$\begin{aligned}
 x_5 &= x_6 + 80 \\
 x_4 &= x_6 - 70
 \end{aligned}$$

$$\begin{aligned}x_3 &= x_6 + 50 \\x_2 &= x_6 \\x_1 &= x_6 + 100\end{aligned}$$

Now in theory, x_6 can take on any value and the equations will be satisfied. However since we're dealing with an English roundabout here, all our x_i 's must be positive, otherwise there would be lots of accidents (cars would be moving in the wrong direction)! For this additional condition to be satisfied, we require $x_6 \geq 70$ in order to make x_4 non-negative. Thus the minimum value of x_6 is 70.