Chapter 8

Eigenvectors, Eigenvalues, and Diagonalization

Section 8.1, p. 420

2. (a) $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$.
   (b) $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$.
   (c) $A\mathbf{x}_3 = \lambda_3\mathbf{x}_3$.

4. $\lambda^2 - 5\lambda + 7$.

6. $p(\lambda) = \lambda^2 - 7\lambda + 6$.

8. $f(\lambda) = \lambda^3$; $\lambda_1 = \lambda_2 = \lambda_3 = 0$; $\mathbf{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}2\\0\\0\end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix}3\\0\\0\end{bmatrix}$.

10. $f(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$; $\lambda_1 = 0$, $\lambda_2 = 2$; $\mathbf{x}_1 = \begin{bmatrix}1\\-1\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}1\\1\end{bmatrix}$.

12. $f(\lambda) = \lambda^3 - 7\lambda^2 + 14\lambda - 8$; $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$; $\mathbf{x}_1 = \begin{bmatrix}-1\\1\\1\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix}7\\-4\\2\end{bmatrix}$.

14. $f(\lambda) = (\lambda - 2)(\lambda + 1)(\lambda - 3)$; $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 3$; $\mathbf{x}_1 = \begin{bmatrix}1\\1\\-4\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

16. (a) $p(\lambda) = \lambda^2 + 1$. The eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix}1\\i\end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix}1\\-i\end{bmatrix}.$$

   (b) $p(\lambda) = \lambda^3 + 2\lambda^2 + 4\lambda + 8$. The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2i$, and $\lambda_3 = -2i$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix}4\\-2\\1\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}-4\\2i\\1\end{bmatrix}, \quad \text{and}\mathbf{x}_3 = \begin{bmatrix}-4\\-2i\\1\end{bmatrix}.$$

(c) $p(\lambda) = \lambda^3 + (-2+i)\lambda^2 - 2i\lambda$. The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -i$, and $\lambda_3 = 2$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

(d) $p(\lambda) = \lambda^2 - 8\lambda + 17$. The eigenvalues are $\lambda_1 = 4+i$ and $\lambda_2 = 4-i$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1+i \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}.$$

18. Basis for eigenspace associated with $\lambda_1 = \lambda_2 = 2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Basis for eigenspace associated with $\lambda_3 = 1$ is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$.

20. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

22. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

24. (a) $\left\{ \begin{bmatrix} -4 \\ 2i \\ 1 \end{bmatrix} \right\}$. (b) $\left\{ \begin{bmatrix} -4 \\ -2i \\ 1 \end{bmatrix} \right\}$.

26. The eigenvalues of A with associated eigenvectors are

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad \lambda_2 = 4, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvalues and associated eigenvectors of

$$A^2 = \begin{bmatrix} 11 & -5 \\ -10 & 6 \end{bmatrix}$$

are

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad \lambda_2 = 16, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

28. $\begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$.

T.1. Let $\mathbf{u}$ and $\mathbf{v}$ be in S, so that $A\mathbf{u} = \lambda_j\mathbf{u}$ and $A\mathbf{v} = \lambda_j\mathbf{v}$. Then

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \lambda_j\mathbf{u} + \lambda_j\mathbf{v} = \lambda_j(\mathbf{u}+\mathbf{v}),$$

so $\mathbf{u}+\mathbf{v}$ is in S. Moreover, if c is any real number, then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda_j\mathbf{u}) = \lambda_j(c\mathbf{u}),$$

so $c\mathbf{u}$ is in S.

T.2. An eigenvector must be a nonzero vector, so the zero vector must be included in S.

T.3. $(\lambda I_n - A)$ is a triangular matrix whose determinant is the product of its diagonal elements, thus the characteristic polynomial of A is

$$f(\lambda) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}).$$

It follows that the eigenvalues of A are the diagonal elements of A.

T.4. $|\lambda I_n - A^T| = |(\lambda I_n - A)^T| = |\lambda I_n - A|$. Associated eigenvectors need not be the same. (But the dimensions of the eigenspace associated with λ, for A and A^T, are equal.)

T.5. $A^k\mathbf{x} = A^{k-1}(A\mathbf{x}) = A^{k-1}(\lambda\mathbf{x}) = \lambda A^{k-1}\mathbf{x} = \cdots = \lambda^k\mathbf{x}$.

T.6. If A is nilpotent and $A^k = O$, and if λ is an eigenvalue for A with associated eigenvector $\mathbf{x}$, then $\mathbf{0} = A^k\mathbf{x} = \lambda^k\mathbf{x}$ implies $\lambda^k = 0$ (since $\mathbf{x} \neq \mathbf{0}$), so $\lambda = 0$.

T.7. (a) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the characteristic polynomial of A. Then

$$f(\lambda) = \det(\lambda I_n - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n).$$

Hence

$$f(0) = \det(-A) = (-\lambda_1)(-\lambda_2) \cdots (-\lambda_n) = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n.$$

Since $\det(-A) = (-1)^n \det(A)$ we have $\det(A) = \lambda_1 \cdots \lambda_n$.

(b) A is singular if and only if for some nonzero vector $\mathbf{x}$, $A\mathbf{x} = \mathbf{0}$, if and only if 0 is an eigenvalue of A. Alternatively, A is singular if and only if $\det(A) = 0$, if and only if [by (a)] 0 is a real root of the characteristic polynomial of A.

T.8. If $A\mathbf{x} = \lambda\mathbf{x}$, $\lambda \neq 0$, then $\lambda^{-1}\mathbf{x} = \lambda^{-1}A^{-1}A\mathbf{x} = \lambda^{-1}A^{-1}(\lambda\mathbf{x}) = \lambda^{-1}\lambda A^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, and thus λ^{-1} is an eigenvalue of A^{-1} with associated eigenvector $\mathbf{x}$.

T.9. (a) The characteristic polynomial of A is

$$\det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{12} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n\,n-1} & \lambda - a_{nn} \end{vmatrix}.$$

Any product in $\det(\lambda I_n - A)$, other than the product of the diagonal entries, can contain at most $n-2$ of the diagonal entries of $\lambda I_n - A$. This follows because at least two of the column indices must be out of natural order in every other product appearing in $\det(\lambda I_n - A)$. This implies that the coefficient of λ^{n-1} is formed by the expansion of the product of the diagonal entries. The coefficient of λ^{n-1} is the sum of the coefficients of λ^{n-1} from each of the products

$$-a_{ii}(\lambda - a_{11}) \cdots (\lambda - a_{i-1\,i-1})(\lambda - a_{i+1\,i+1}) \cdots (\lambda - a_{nn})$$

$i = 1, 2, \ldots, n$. The coefficient of λ^{n-1} in each such term is $-a_{ii}$, so the coefficient of λ^{n-1} in the characteristic polynomial is

$$-a_{11} - a_{22} - \cdots - a_{nn} = -\operatorname{Tr}(A).$$

(b) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then $\lambda - \lambda_i$, $i = 1, 2, \ldots, n$ are factors of the characteristic polynomial $\det(\lambda I_n - A)$. It follows that

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Proceeding as in (a), the coefficient of λ^{n-1} is the sum of the coefficients of λ^{n-1} from each of the products

$$-\lambda_i(\lambda - \lambda_1) \cdots (\lambda - \lambda_{i-1})(\lambda - \lambda_{i+1}) \cdots (\lambda - \lambda_n)$$

for $i = 1, 2, \ldots, n$. The coefficient of λ^{n-1} in each such term is $-\lambda_i$, so the coefficient of λ^{n-1} in the characteristic polynomial is $-\lambda_1 - \lambda_2 - \cdots - \lambda_n = -\operatorname{Tr}(A)$ by (a). Thus, $\operatorname{Tr}(A)$ is the sum of the eigenvalues of A.

(c) We have

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

so the constant term is $\pm\lambda_1\lambda_2 \cdots \lambda_n$.

(d) If $f(\lambda) = \det(\lambda I_n - A)$ is the characteristic polynomial of A, then $f(0) = \det(-A) = (-1)^n \det(A)$. Since $f(0) = a_n$, the constant term of $f(\lambda)$, $a_n = (-1)^n \det(A)$. The result follows from part (c).

T.10. Suppose there is a vector $\mathbf{x} \neq \mathbf{0}$ in both S_1 and S_2. Then $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$. So $(\lambda_2 - \lambda_1)\mathbf{x} = \mathbf{0}$. Hence $\lambda_1 = \lambda_2$ since $\mathbf{x} \neq \mathbf{0}$, a contradiction. Thus the zero vector is the only vector in both S_1 and S_2.

T.11. If $A\mathbf{x} = \lambda\mathbf{x}$, then, for any scalar r,

$$(A + rI_n)\mathbf{x} = A\mathbf{x} + r\mathbf{x} = \lambda\mathbf{x} + r\mathbf{x} = (\lambda + r)\mathbf{x}.$$

Thus $\lambda + r$ is an eigenvalue of $A + rI_n$ with associated eigenvector $\mathbf{x}$.

T.12. (a) Since $A\mathbf{u} = \mathbf{0} = 0\mathbf{0}$, it follows that 0 is an eigenvalue of A with associated eigenvector $\mathbf{u}$.

(b) Since $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, it follows that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, namely $\mathbf{x} = \mathbf{v}$.

T.13. We have

(a) $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x} = (\lambda + \mu)\mathbf{x}$.

(b) $(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mu\mathbf{x}) = \mu(A\mathbf{x}) = (\mu\lambda)\mathbf{x} = (\lambda\mu)\mathbf{x}$.

T.14. Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The product

$$A^T\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{x} = 1\mathbf{x}$$

so $\lambda = 1$ is an eigenvalue of A^T. By Exercise T.4, $\lambda = 1$ is also an eigenvalue of A.

T.15. Let W be the eigenspace of A with associated eigenvalue λ. Let $\mathbf{w}$ be in W. Then $L(\mathbf{w}) = A\mathbf{w} = \lambda\mathbf{w}$. Therefore $L(\mathbf{w})$ is in W since W is closed under scalar multiplication.

ML.1. Enter each matrix A into MATLAB and use command **poly(A)**.

(a) **A = [1 2; − 1 4];**
M = (3 * eye(size(A)) − A)
rref([M [0 0]′])
ans =

1	−1	0
0	0	0

The general solution is $x_1 = x_2$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & 1\end{bmatrix}'$ is an eigenvector.

(b) **A = [4 0 0;1 3 0;2 1 − 1];**
M = (− 1 * eye(size(A)) − A)
rref([M [0 0 0]′])
ans =

1	0	0	0
0	1	0	0
0	0	0	0

The general solution is $x_3 = r$, $x_2 = 0$, $x_1 = 0$. Let $r = 1$ and we have that $\begin{bmatrix}0 & 0 & 1\end{bmatrix}'$ is an eigenvector.

(c) **A = [2 1 2;2 2 − 2;3 1 1];**
M = (2 * eye(size(A)) − A)
rref([M [0 0 0]′])
ans =

1	0	−1	0
0	1	2	0
0	0	0	0

The general solution is $x_3 = r$, $x_2 = -2x_3 = -2r$, $x_1 = x_3 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & -2 & 1\end{bmatrix}'$ is an eigenvector.

ML.4. Approximately $\begin{bmatrix}1.0536\\-0.47\\-0.37\end{bmatrix}$.

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2. Not diagonalizable. The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$. Associated eigenvectors are $\mathbf{x}_1 = \mathbf{x}_2 = r\begin{bmatrix}0\\1\end{bmatrix}$, where r is any nonzero real number.

4. Diagonalizable. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$. The result follows by Theorem 8.5.

6. Diagonalizable. The eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = 3$. Associated eigenvectors are, respectively,
$$\mathbf{x}_1 = \begin{bmatrix}-1\\1\end{bmatrix} \quad\text{and}\quad \mathbf{x}_2 = \begin{bmatrix}2\\5\end{bmatrix}.$$

8. Not diagonalizable. The eigenvalues of A are $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 5$. An eigenvector associated with λ_1 is
$$\mathbf{x}_1 = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}.$$

eigenvectors associated with $\lambda_2 = \lambda_3$ are

$$r\begin{bmatrix} -2 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

Since we cannot find two linearly independent eigenvectors associated with $\lambda_2 = \lambda_3$ we conclude that A is not diagonalizable.

10. $\begin{bmatrix} 3 & 5 & -5 \\ 5 & 3 & -5 \\ 5 & 5 & -7 \end{bmatrix}$.

12. $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 3$. Associated eigenvectors are the columns of P. (P is not unique.)

14. $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$. Associated eigenvectors are the columns of P. (P is not unique.)

16. Not possible.

18. $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 6$. Associated eigenvectors are the columns of P. (P is not unique.)

20. Not possible.

22. Not possible.

24. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

26. $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$.

28. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Other answers are possible.

30. No.

32. No.

34. The eigenvalues of the given matrix are $\lambda_1 = 0$ and $\lambda_2 = 7$. By Theorem 8.5, the given matrix is diagonalizable. $D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ is similar to the given matrix.

36. The eigenvalues of the given matrix are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$. Associated eigenvectors are, respectively,

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

By Theorem 8.4, the given matrix is diagonalizable. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is similar to the given matrix.

38. A is upper triangular with multiple eigenvalue $\lambda_1 = \lambda_2 = 2$ and associated eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

40. A has the multiple eigenvalue $\lambda_1 = \lambda_2 = 2$ with associated eigenvector $\begin{bmatrix} -3 \\ -7 \\ 8 \\ 0 \end{bmatrix}$.

42. Not defective.

44. Not defective.

46. $\begin{bmatrix} 768 & -1280 \\ 256 & -768 \end{bmatrix}$.

T.1. (a) $A = P^{-1}AP$ for $P = I_n$.
(b) If $B = P^{-1}AP$, then $A = PBP^{-1}$ and so A is similar to B.
(c) If $B = P^{-1}AP$ and $C = Q^{-1}BQ$ then $C = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$ with PQ nonsingular.

T.2. If A is diagonalizable, then there is a nonsingular matrix P so that $P^{-1}AP = D$, a diagonal matrix. Then $A^{-1} = PD^{-1}P^{-1} = (P^{-1})^{-1}D^{-1}P^{-1}$. Since D^{-1} is a diagonal matrix, we conclude that A^{-1} is diagonalizable.

T.3. Necessary and sufficient conditions are: $(a-d)^2 + 4bc > 0$ for $b = c = 0$.
For the characteristic polynomial of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad f(\lambda) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 + \lambda(-a-d) + ad - bc.$$

Then $f(\lambda)$ has real roots if and only if $(a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc \geq 0$. If $(a-d)^2 + 4bc > 0$, then the eigenvalues are distinct and we can diagonalize. On the other hand, if $(a-d)^2 + 4bc = 0$, then the two eigenvalues λ_1 and λ_2 are equal and we have $\lambda_1 = \lambda_2 = \frac{a+d}{2}$. To find associated eigenvectors we solve the homogeneous system

$$\begin{bmatrix} \frac{d-a}{2} & -b \\ -c & \frac{a-d}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this case A is diagonalizable if and only if the solution space has dimension $= 2$; that is, if and only if the rank of the coefficient matrix $= 0$, thus, if and only if $b = c = 0$ so that A is already diagonal.

T.4. We show that the characteristic polynomials of AB^{-1} and $B^{-1}A$ are the same. The characteristic polynomial of AB^{-1} is

$$\begin{aligned} f(\lambda) = |\lambda I_n - AB^{-1}| &= |\lambda BB^{-1} - AB^{-1}| \\ &= |(\lambda B - A)B^{-1}| = |\lambda B - A|\,|B^{-1}| = |B^{-1}|\,|\lambda B - A| \\ &= |B^{-1}(\lambda B - A)| = |\lambda B^{-1}B - B^{-1}A| = |\lambda I_n - B^{-1}A|, \end{aligned}$$

which is the charcteristic polynomial of $B^{-1}A$.

T.5. $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1$, but all the eigenvectors are of the form $r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Clearly A has only one linearly independent eigenvector and is not diagonalizable. However, $\det(A) \neq 0$, so A is nonsingular. (See also Example 6 in Section 8.2.)

T.6. We have $BA = A^{-1}(AB)A$, so AB and BA are similar. By Theorem 8.3, AB and BA have the same eigenvalues.

T.7. (a) If $P^{-1}AP = D$, a diagonal matrix, then $P^TA^T(P^{-1})^T = (P^{-1}AP)^T = D^T$ is diagonal, and $P^T = ((P^{-1})^T)^{-1}$, so A is similar to a diagonal matrix.

(b) $P^{-1}A^kP = (P^{-1}AP)^k = D^k$ is diagonal.

T.8. Suppose that A and B are similar, so that $B = P^{-1}AP$. Then it follows that $B^k = P^{-1}A^kP$, for any nonnegative integer k. Hence, A^k and B^k are similar.

T.9. Suppose that A and B are similar, so that $B = P^{-1}AP$. Then

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P) = \det(A).$$

T.10. We have $B = P^{-1}AP$ and $A\mathbf{x} = \lambda\mathbf{x}$. Therefore $BP^{-1} = P^{-1}APP^{-1} = P^{-1}A$ and hence

$$B(P^{-1}\mathbf{x}) = (BP^{-1})\mathbf{x} = P^{-1}A\mathbf{x} = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$$

which shows that $P^{-1}\mathbf{x}$ is an eigenvector of B associated with the eigenvalue λ.

T.11. The proof proceeds as in the proof of Theorem 8.5, with $k = n$.

T.12. The result follows at once from Theorems 8.2 and 8.3.

ML.1. (a) **A = [0 2; − 1 3];**
r = roots(poly(A))

```
r =
    2
    1
```

The eigenvalues are distinct so A is diagonalizable. We find the corresponding eigenvectors.
M = (2 ∗ eye(size(A)) − A)
rref([M [0 0]'])

```
ans =
      1  -1   0
      0   0   0
```

The general solution is $x_1 = x_2$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & 1\end{bmatrix}'$ is an eigenvector.
M = (1 ∗ eye(size(A)) − A)
rref([M [0 0]'])

```
ans =
      1  -2   0
      0   0   0
```

The general solution is $x_1 = 2x_2$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}2 & 1\end{bmatrix}'$ is an eigenvector.
P = [1 1;2 1]'

```
P =
    1  2
    1  1
```

invert(P) ∗ A ∗ P

```
ans =
      2  0
      0  1
```