8. $f(x,y) = y\ln x$

(a) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle y/x, \ln x\rangle$ (b) $\nabla f(1,-3) = \left\langle \frac{-3}{1}, \ln 1\right\rangle = \langle -3, 0\rangle$

(c) By Equation 9, $D_{\mathbf{u}}\, f(1,-3) = \nabla f(1,-3)\cdot \mathbf{u} = \langle -3, 0\rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle = \frac{12}{5}$.

10. $f(x,y,z) = \sqrt{x+yz} = (x+yz)^{1/2}$

(a) $\nabla f(x,y,z) = \left\langle \frac{1}{2}(x+yz)^{-1/2}(1), \frac{1}{2}(x+yz)^{-1/2}(z), \frac{1}{2}(x+yz)^{-1/2}(y)\right\rangle$

$$= \langle 1/\left(2\sqrt{x+yz}\right), z/\left(2\sqrt{x+yz}\right), y/\left(2\sqrt{x+yz}\right)\rangle$$

(b) $\nabla f(1,3,1) = \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right\rangle$

(c) $D_{\mathbf{u}} f(1,3,1) = \nabla f(1,3,1)\cdot \mathbf{u} = \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right\rangle \cdot \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle = \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}$.

20. $f(p,q) = qe^{-p} + pe^{-q} \quad \Rightarrow \quad \nabla f(p,q) = \left\langle -qe^{-p} + e^{-q}, e^{-p} - pe^{-q}\right\rangle$.

$\nabla f(0,0) = \langle 1, 1\rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(0,0)| = \sqrt{2}$.

30. $z = f(x,y) = 1000 - 0.005x^2 - 0.01y^2 \quad \Rightarrow \quad \nabla f(x,y) = \langle -0.01x, -0.02y\rangle$ and $\nabla f(60,40) = \langle -0.6, -0.8\rangle$

(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and $D_{\mathbf{u}} f(60,40) = \nabla f\,(60,40)\cdot\langle 0,-1\rangle = \langle -0.6, -0.8\rangle \cdot \langle 0, -1\rangle = 0.8$. Thus, if you walk due south from $(60, 40, 966)$ you will ascend at a rate of 0.8 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1, 1\rangle$ and $D_{\mathbf{u}} f(60,40) = \nabla f\,(60,40)\cdot \frac{1}{\sqrt{2}}\langle -1,1\rangle = \langle -0.6,-0.8\rangle \cdot \frac{1}{\sqrt{2}}\langle -1, 1\rangle = -\frac{0.2}{\sqrt{2}} \approx -0.14$. Thus, if you walk northwest from $(60, 40, 966)$ you will descend at a rate of approximately 0.14 vertical meters per horizontal meter.

(c) $\nabla f(60,40) = \langle -0.6, -0.8\rangle$ is the direction of largest slope with a rate of ascent $|\nabla f(60,40)| = \sqrt{(-0.6)^2 + (-0.8)^2} = 1$. The angle above the horizontal in which the path begins is given by $\tan\theta = 1 \quad \Rightarrow \quad \theta = 45^\circ$.

2. (a) $D = g_{xx}(0,2)\,g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0,2)\,g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0,2)\,g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

Name: ___________________ Class:___ Date: ______________

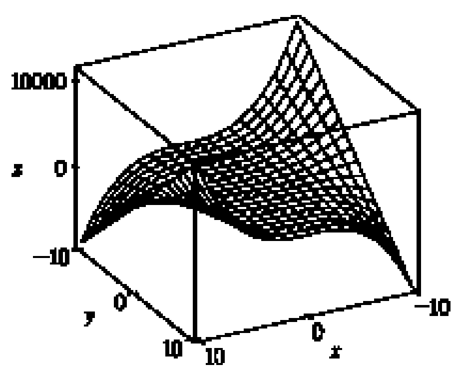
4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x,y)=3x-x^3-2y^2+y^4 \;\Rightarrow\; f_x(x,y)=3-3x^2$, $f_y(x,y)=-4y+4y^3$. Setting these partial derivatives equal to 0, we have $3-3x^2=0 \;\Rightarrow\; x=\pm1$ and $-4y+4y^3=0 \;\Rightarrow\; y(y^2-1)=0 \;\Rightarrow\; y=0,\pm1$. So our critical points are $(\pm1,0)$, $(\pm1,\pm1)$. The second partial derivatives are $f_{xx}(x,y)=-6x$, $f_{xy}(x,y)=0$, and $f_{yy}(x,y)=12y^2-4$, so $D(x,y)=f_{xx}(x,y)\,f_{yy}(x,y)-[f_{xy}(x,y)]^2=(-6x)(12y^2-4)-(0)^2=-72xy^2+24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D>0,\ f_{xx}<0 \;\Rightarrow\; f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D<0 \;\Rightarrow\; f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D<0 \;\Rightarrow\; f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D<0 \;\Rightarrow\; f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D>0,\ f_{xx}>0 \;\Rightarrow\; f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D>0,\ f_{xx}>0 \;\Rightarrow\; f$ has a local minimum at $(-1,-1)$

6. $f(x,y)=x^3y+12x^2-8y \;\Rightarrow\; f_x=3x^2y+24x$, $f_y=x^3-8$, $f_{xx}=6xy+24$, $f_{xy}=3x^2$, $f_{yy}=0$. Then $f_y=0$ implies $x=2$, and substitution into $f_x=0$ gives $12y+48=0 \;\Rightarrow\; y=-4$. Thus, the only critical point is $(2,-4)$.

$D(2,-4)=(-24)(0)-12^2=-144<0$, so $(2,-4)$ is a saddle point.



10. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow \quad f_x = 6x^2 + y^2 + 10x$,

$f_y = 2xy + 2y$, $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$. Then

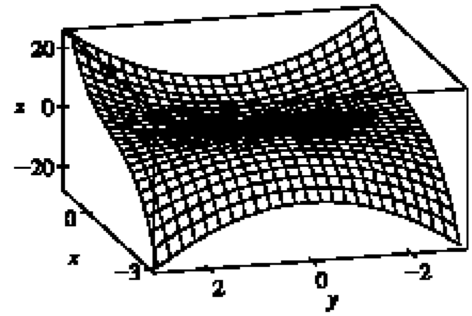
$f_y = 0$ implies $y = 0$ or $x = -1$. Substituting into $f_x = 0$ gives the

critical points $(0,0)$, $\left(-\frac{5}{3}, 0\right)$, $(-1, \pm 2)$. Now $D(0,0) = 20 > 0$ and

$f_{xx}(0,0) = 10 > 0$, so $f(0,0) = 0$ is a local minimum. Also

$f_{xx}\left(-\frac{5}{3}, 0\right) < 0$, $D\left(-\frac{5}{3}, 0\right) > 0$, and $D(-1, \pm 2) < 0$. Hence

$f\left(-\frac{5}{3}, 0\right) = \frac{125}{27}$ is a local maximum while $(-1, \pm 2)$ are saddle points.



45. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\ \text{m}^3$. Then

$f(x,y) = xy + [64{,}000(x+y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$

implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now

$D(x,y) = [(2)(64{,}000)]^2 x^{-3} y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40,40) > 0$ so this is indeed a minimum. Thus the

dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

4. $f(x,y) = 4x + 6y$, $g(x,y) = x^2 + y^2 = 13 \quad \Rightarrow \quad \nabla f = \langle 4, 6 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply

$x = \dfrac{2}{\lambda}$ and $y = \dfrac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\dfrac{2}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 \quad \Rightarrow \quad 13 = \dfrac{13}{\lambda^2} \quad \Rightarrow \quad \lambda = \pm 1$, so f has possible extreme values

at the points $(2,3)$, $(-2,-3)$. We compute $f(2,3) = 26$ and $f(-2,-3) = -26$, so the maximum value of f on

$x^2 + y^2 = 13$ is $f(2,3) = 26$ and the minimum value is $f(-2,-3) = -26$.

12. $f(x,y,z) = x^4 + y^4 + z^4$, $g(x,y,z) = x^2 + y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 4x^3, 4y^3, 4z^3 \rangle$, $\lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$.

Case 1: If $x \neq 0, y \neq 0$ and $z \neq 0$ then $\nabla f = \lambda \nabla g$ implies $\lambda = 2x^2 = 2y^2 = 2z^2$ or $x^2 = y^2 = z^2 = \frac{1}{3}$ yielding 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1. Thus on $x^2 + y^2 + z^2 = 1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

18. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \quad \Rightarrow \quad \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad x = 1, y = 0$. Thus $(1,0)$ is the only critical point

of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \quad \Rightarrow \quad \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so

$6y = 2\lambda y \quad \Rightarrow \quad$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad \Rightarrow \quad x = -2$ and

$y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of

$f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.