(b) Possible answers: $\begin{bmatrix}0\\4\\4\end{bmatrix}$, $\begin{bmatrix}1\\2\\0\end{bmatrix}$.

T.1. (a) $f(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = f(\mathbf{u}) + f(\mathbf{v})$.

(b) $f(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cf(\mathbf{u})$.

(c) $f(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}) + A(c\mathbf{v}) = c(A\mathbf{u}) + d(A\mathbf{v}) = cf(\mathbf{u}) + df(\mathbf{v})$.

T.2. For any real numbers c and d, we have

$$f(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u})+A(d\mathbf{v}) = c(A\mathbf{u})+d(A\mathbf{v}) = cf(\mathbf{u})+df(\mathbf{v}) = c\mathbf{0}+d\mathbf{0} = \mathbf{0}+\mathbf{0} = \mathbf{0}.$$

T.3. (a) $O(\mathbf{u}) = \begin{bmatrix}0 & \cdots & 0\\ & \vdots & \\ 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} = \mathbf{0}.$

(b) $I(\mathbf{u}) = \begin{bmatrix}1 & 0 & \cdots & 0\\ & & \vdots & \\ 0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \mathbf{u}.$

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2. Neither.

4. Neither.

6. Neither.

8. Neither.

10. (a) $\begin{bmatrix}2 & 0 & 4 & 2\\ -1 & 3 & 1 & 1\\ 3 & -2 & 5 & 6\end{bmatrix}$.

(b) $\begin{bmatrix}2 & 0 & 4 & 2\\ -12 & 8 & -20 & -24\\ -1 & 3 & 1 & 1\end{bmatrix}$.

(c) $\begin{bmatrix}0 & 6 & 6 & 4\\ 3 & -2 & 5 & 6\\ -1 & 3 & 1 & 1\end{bmatrix}$.

12. Possible answers:

(a) $\begin{bmatrix}4 & 3 & 7 & 5\\ 2 & 0 & 1 & 4\\ -2 & 4 & -2 & 6\end{bmatrix}$.

(b) $\begin{bmatrix}3 & 5 & 6 & 8\\ -4 & 8 & -4 & 12\\ 2 & 0 & 1 & 4\end{bmatrix}$.

(c) $\begin{bmatrix}4 & 3 & 7 & 5\\ -1 & 2 & -1 & 3\\ 0 & 4 & -1 & 10\end{bmatrix}$.

14. $\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

16. $\begin{bmatrix} 1 & -2 & 1 & 4 & -3 \\ 0 & 1 & -\frac{2}{3} & -\frac{7}{3} & \frac{10}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$

18. (a) No. (b) Yes. (c) Yes. (d) No.

20. (a) $x = 1, y = 2, z = -2$. (b) No solution. (c) $x = 1, y = 1, z = 0$.
(d) $x = 0, y = 0, z = 0$.

22. (a) $x = 1 - \frac{2}{5}r,\ y = -1 + \frac{1}{5}r,\ z = r$. (b) $x = 1 - r,\ y = 3 + r,\ z = 2 - r,\ w = r$. (c) No solution.
(d) $x = 0, y = 0, z = 0$.

24. (a) $a = \pm\sqrt{3}$. (b) $a \neq \pm\sqrt{3}$. (c) None.

26. (a) $a = -3$. (b) $a \neq \pm 3$. (c) $a = 3$.

28. (a) $x = r, y = -2r, z = r, r =$ any real number. (b) $x = 1, y = 2, z = 2$.

30. (a) No solution. (b) $x = 1 - r, y = 2 + r, z = -1 + r, r =$ any real number.

32. $x = -2 + r, y = 2 - 2r, z = r$, where r is any real number.

34. $c - b - a = 0$

36. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}.$

38. $x = 5r, y = 6r, z = r, r =$ any nonzero real number.

40. $-a + b - c = 0$.

42. $\mathbf{x} = \begin{bmatrix} r \\ r \end{bmatrix}, r \neq 0.$

44. $\mathbf{x} = \begin{bmatrix} -\frac{1}{2}r \\ \frac{1}{2}r \\ r \end{bmatrix}, r \neq 0.$

46. $\mathbf{x} = \begin{bmatrix} \frac{19}{6} \\ -\frac{59}{30} \\ \frac{17}{30} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}r \\ \frac{2}{15}r \\ \frac{19}{15}r \\ r \end{bmatrix}.$

48. $y = \frac{25}{2}x^2 - \frac{61}{2}x + 23$.

50. $y = \frac{2}{3}x^3 + \frac{4}{3}x^2 - \frac{2}{3}x + \frac{2}{3}$.

52. 60 in deluxe binding. If r is the number in bookclub binding, then r is an integer which must satisfy $0 \leq r \leq 90$ and then the number of paperbacks is $180 - 2r$.

54. $\frac{3}{2}x^2 - x + \frac{1}{2}$.

56. (a) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

58. (a) Inconsistent.

(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

T.1. Suppose the leading one of the ith row occurs in the jth column. Since leading ones of rows $i+1, i+2, \ldots$ are to the right of that of the ith row, and in any nonzero row, the leading one is the first nonzero element, all entries in the jth column below the ith row must be zero.

T.2. (a) A is row equivalent to itself: the sequence of operations is the empty sequence.

(b) Each elementary row operation of types (a), (b) or (c) has a corresponding inverse operation of the same type which "undoes" the effect of the original operation. For example, the inverse of the operations "add d times row r of A to row s of A" is "add $-d$ times row r of A to row s of A." Since B is assumed row equivalent to A, there is a sequence of elementary row operations which gets from A to B. Take those operations in the reverse order, and for each operation do its inverse, and that takes B to A. Thus A is row equivalent to B.

(c) Follow the operations which take A to B with those which take B to C.

T.3. The sequence of elementary row operations which takes A to B, when applied to the augmented matrix $\left[A \mid \mathbf{0}\right]$, yields the augmented matrix $\left[B \mid \mathbf{0}\right]$. Thus both systems have the same solutions, by Theorem 1.7.

T.4. A linear system whose augmented matrix has the row

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \vdots & 1 \end{bmatrix} \tag{1.2}$$

can have no solution: that row corresponds to the unsolvable equation $0x_1 + 0x_2 + \cdots + 0x_n = 1$. If the augmented matrix of $A\mathbf{x} = \mathbf{b}$ is row equivalent to a matrix with the row (1.2) above, then by Theorem 1.7, $A\mathbf{x} = \mathbf{b}$ can have no solution.

Conversely, assume $A\mathbf{x} = \mathbf{b}$ has no solution. Its augmented matrix is row equivalent to some matrix $\left[C \mid D\right]$ in reduced row echelon form. If $\left[C \mid D\right]$ does not contain the row (1.2) then it has at most m nonzero rows, and the leading entries of those nonzero rows all correspond to unknowns of the system. After assigning values to the free variables — the variables not corresponding to leading entries of rows — one gets a solution to the system by solving for the values of the leading entry variables. This contradicts the assumption that the system had no solution.

T.5. If $ad - bc = 0$, the two rows of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are multiples of one another:

$$c\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} ac & bc \end{bmatrix} \quad \text{and} \quad a\begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \end{bmatrix} \quad \text{and} \quad bc = ad.$$

Any elementary row operation applied to A will produce a matrix with rows that are multiples of each other. In particular, elementary row operations cannot produce I_2, and so I_2 is not row equivalent to A. If $ad - bc \neq 0$, then a and c are not both 0. Suppose $a \neq 0$.

a	b	1	0	Multiply the first row by $\frac{1}{a}$, and add $(-c)$ times the first row to the second row.
c	d	0	1	
1	$\dfrac{b}{a}$	$\dfrac{1}{a}$	0	Multiply the second row by $\frac{a}{ad-bc}$.
0	$d - \dfrac{bc}{a}$	$-\dfrac{c}{a}$	1	
1	$\dfrac{b}{a}$	$\dfrac{1}{a}$	0	Add $\left(-\frac{b}{a}\right)$ times the second row to the first row.
0	1	$\dfrac{-c}{ad-bc}$	$\dfrac{1}{ad-bc}$	
1	0	$\dfrac{d}{ad-bc}$	$\dfrac{-b}{ad-bc}$	
0	1	$\dfrac{-c}{ad-bc}$	$\dfrac{a}{ad-bc}$	

T.6. (a) Since $a(kb) - b(ka) = 0$, it follows from Exercise T.5 that A is not row equivalent to I_2.

(b) Suppose that $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$. Since $0 \cdot b - 0 \cdot c = 0$, it follows from Exercise T.5 that A is not row equivalent to I_2.

T.7. For any angle θ, $\cos\theta$ and $\sin\theta$ are not both zero. Assume that $\cos\theta \neq 0$ and proceed as follows. The row operation $\frac{1}{\cos\theta}$ times row 1 gives

$$\begin{bmatrix} 1 & \dfrac{\sin\theta}{\cos\theta} \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Applying row operation $\sin\theta$ times row 1 added to row 2 we obtain

$$\begin{bmatrix} 1 & \dfrac{\sin\theta}{\cos\theta} \\ 0 & \cos\theta + \dfrac{\sin^2\theta}{\cos\theta} \end{bmatrix}.$$

Simplifying the $(2,2)$-entry we have

$$\cos\theta + \frac{\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta}$$

x = A\b

x =

```
1  0  -1  0
0  1   2  0
0  0   0  1
```

This augmented matrix implies that the system is inconsistent. We can also infer that the coefficient matrix is singular.

```
x = A\b
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate.
RCOND=2.937385e-018.
x =
  1.0e + 015*
     3.1522
    -6.3044
     3.1522
```

Each element of the solution displayed using \ is huge. This, together with the warning, suggests that errors due to using computer arithmetic were magnified in the solution process. MATLAB uses an LU-factorization procedure when \ is used to solve linear systems (see Section 1.7), while **rref** actually rounds values before displaying them.

ML.16. $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \begin{array}{rl} x_1\ x_2\ x_3 + x_4 & = 0 \\ x_2\ x_3 + x_4 + x_5 & = 1 \\ x_3 + x_4 & = 1 \end{array} \Longrightarrow \begin{array}{l} x_1 = x_4 \\ x_2 = 1 + x_4 + x_5 \\ x_3 = 1 + x_4 \\ x_4 = x_4 \\ x_5 = x_5 \end{array} \Longrightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

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2. Adding twice the first row to the second row produces a row of zeros.

4. Singular.

6. (a) Singular. (b) $\begin{bmatrix} 1 & -1 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ -1 & 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{5} & 1 & \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}$.

8. (a) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$. (b) $\begin{bmatrix} 3 & 2 & -4 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. (c) Singular.

10. (a) $\begin{bmatrix} \frac{3}{5} & -\frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{3}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$. (b) Singular. (c) Singular.

12. (b) and (c).

14. $\begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$.

15. If the jth column A_j of A consists entirely of zeros, then so does the jth column BA_j of BA (Exercise T.9(a), Sec. 1.3), so A is singular. If the ith row A_i of A consists entirely of zeros, then for any B, the ith row A_iB of AB is zero, so again A is singular.

16. $a \neq 0$, $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{bmatrix}$.

18. (a) $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$. (b) $(A^T)^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = (A^{-1})^T$.

19. Yes. $(A^{-1})^T A = (A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$. By Theorem 1.9, $(A^{-1})^T = A^{-1}$. That is, A^{-1} is symmetric.

20. (a) No. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $(A+B)^{-1}$ exists but A^{-1} and B^{-1} do not. Even supposing they all exist, equality need not hold. For example, let $A = [1]$, $B = [2]$. Then $(A+B)^{-1} = \left[\frac{1}{3}\right] \neq [1] + \left[\frac{1}{2}\right] = A^{-1} + B^{-1}$.

(b) Yes for A nonsingular and $c \neq 0$.

$$(cA)\left(\frac{1}{c}A^{-1}\right) = c\left(\frac{1}{c}\right) A \cdot A^{-1} = 1 \cdot I_n = I_n.$$

22. $A + B$ may be singular: let $A = I_n$ and $B = -I_n$.
$A - B$ may be singular: let $A = B = I_n$.
$-A$ is nonsingular: $(-A)^{-1} = -(A^{-1})$.

24. $\begin{bmatrix} 11 & 19 \\ 7 & 0 \end{bmatrix}$.

26. Singular. Since the given homogeneous system has a nontrivial solution, Theorem 1.12 implies that A is singular.

28. $\left[\begin{array}{cc:cc} 3 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ \hdashline 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & -6 \end{array}\right]$.

30. (a) Singular. (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. (c) Singular.

32. (a) No. (b) Yes.

T.1. B is nonsingular, so B^{-1} exists, and

$$A = AI_n = A(BB^{-1}) = (AB)B^{-1} = OB^{-1} = O.$$

T.2. The case $r = 2$ of Corollary 1.2 is Theorem 1.10(b). In general, if $r > 2$,

$$\begin{aligned}(A_1A_2\cdots A_r)^{-1} &= [(A_1A_2\cdots A_{r-1})A_r]^{-1}\\ &= A_r^{-1}(A_1A_2\cdots A_{r-1})^{-1}\\ &= A_r^{-1}\left[(A_1A_2\cdots A_{r-2})A_{r-1}\right]^{-1}\\ &= A_r^{-1}A_{r-1}^{-1}(A_1A_2\cdots A_{r-2})^{-1}\\ &= \cdots = A_r^{-1}A_{r-1}^{-1}\cdots A_1^{-1}.\end{aligned}$$

T.3. A is row equivalent to a matrix B in reduced row echelon form which, by Theorem 1.11 is not I_n. Thus B has fewer than n nonzero rows, and fewer than n unknowns corresponding to pivotal columns of B. Choose one of the free unknowns — unknowns not corresponding to pivotal columns of B. Assign any nonzero value to that unknown. This leads to a nontrivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

T.4. The result follows from Theorem 1.12 and Exercise T.8 of Section 1.5.

T.5. For any angle θ, $\cos\theta$ and $\sin\theta$ are never simultaneously zero. Thus at least one element in column 1 is not zero. Assume $\cos\theta \neq 0$. (If $\cos\theta = 0$, then interchange rows 1 and 2 and proceed in a similar manner to that described below.) To show that the matrix is nonsingular and determine its inverse, we put

$$\left[\begin{array}{cc:cc}\cos\theta & \sin\theta & 1 & 0\\ -\sin\theta & \cos\theta & 0 & 1\end{array}\right]$$

into reduced row echelon form. Apply row operations $\frac{1}{\cos\theta}$ times row 1 and $\sin\theta$ times row 1 added to row 2 to obtain

$$\left[\begin{array}{cc:cc}1 & \dfrac{\sin\theta}{\cos\theta} & \dfrac{1}{\cos\theta} & 0\\ 0 & \dfrac{\sin^2\theta}{\cos\theta} + \cos\theta & \dfrac{\sin\theta}{\cos\theta} & 1\end{array}\right].$$

Since

$$\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta},$$

the $(2,2)$-element is not zero. Applying row operations $\cos\theta$ times row 2 and $\left(-\frac{\sin\theta}{\cos\theta}\right)$ times row 2 added to row 1 we obtain

$$\left[\begin{array}{cc:cc}1 & 0 & \cos\theta & -\sin\theta\\ 0 & 1 & \sin\theta & \cos\theta\end{array}\right].$$

It follows that the matrix is nonsingular and its inverse is

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

T.6. Let $A = [a_{ij}]$ be a nonsingular upper triangular matrix, where $a_{ij} = 0$ for $i > j$. We seek a matrix $B = [b_{ij}]$ such that $AB = I_n$ and $BA = I_n$. Using the equation $BA = I_n$, we find that $\sum_{k=1}^{n} b_{ik}a_{ki} = 1$, and since $a_{ki} = 0$ for $k > i$, this equation reduces to $b_{ii}a_{ii} = 1$. Thus, we must have $a_{ii} \neq 0$ and $b_{ii} = 1/a_{ii}$. The equation $\sum_{k=1}^{n} b_{ik}a_{kj} = 0$ for $i \neq j$ implies that $b_{ij} = 0$ for $i > j$. Hence, $B = A^{-1}$ is upper triangular.

T.7. Let $\mathbf{u}$ be one solution to $A\mathbf{x} = \mathbf{b}$. Since A is singular, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{u}_0$. Then for any real number r, $\mathbf{v} = r\mathbf{u}_0$ is also a solution to the homogeneous system. Finally, by Exercise T.13(a), Sec. 1.5, for each of the infinitely many matrices $\mathbf{v}$, the matrix $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is a solution to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$.

T.8. Let A be nonsingular and symmetric. We have $(A^{-1})^T = (A^T)^{-1} = A^{-1}$, so A^{-1} is symmetric.

T.9. Let $A = [a_{ij}]$ be a diagonal matrix with nonzero diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$. That is, $a_{ij} \neq 0$ if $i = j$ and 0 otherwise. We seek an $n \times n$ matrix $B = [b_{ij}]$ such that $AB = I_n$. The (i, j) entry in AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$, so $\sum_{k=1}^{n} a_{ik}b_{kj} = 1$ if $i = j$ and 0 otherwise. This implies that $b_{ii} = 1/a_{ii}$ and $b_{ij} = 0$ if $i \neq j$. Hence, A is nonsingular and $A^{-1} = B$.

T.10. $B^k = PA^kP^{-1}$.

T.11. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$

T.12. No, because if $AB = O$, then $A^{-1}AB = B = A^{-1}O = O$, which contradicts that B is nonsingular.

T.13. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}.$$

Thus $A^2 = O$ provided

$$\begin{aligned} ab + bd &= b(a + d) = 0 \\ ac + cd &= c(a + d) = 0 \\ a^2 + bc &= 0 \\ d^2 + bc &= 0 \end{aligned}$$

Case $b = 0$. Then $a^2 = 0 \Longrightarrow a = 0$ and $d^2 = 0 \Longrightarrow d = 0$. But $bc = 0$. Hence b could be either 1 or 0. So

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Case $c = 0$. Similarly $a = d = 0$ and $c = 1$ or 0. So

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Case $a + d = 0 \Longrightarrow a = d = 0$ or $a = d = 1$

(i) $a = d = 0 \Longrightarrow bc = 0$ so we have $c = b = 0$ or $c = 0$, $b = 1$ or $c = 1$, $b = 0$. Thus

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$