where

$$R = \frac{R_1 + R_2}{R_1 R_2},$$

so

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

T.2. We choose the following directions for the currents:

$$\begin{aligned} I &: a \text{ to } b \\ I' &: b \text{ to } c \\ I_1 &: b \text{ to } g \\ I_2 &: c \text{ to } f \\ I_3 &: d \text{ to } e \end{aligned}$$

Then we have the following linear equations:

$$\begin{aligned} I - I' - I_1 &= 0 && \text{(node } b) \\ I' - I_2 - I_3 &= 0 && \text{(node } c) \\ -I_1R_1 + I_2R_2 &= 0 && \text{(loop } bgfcb) \\ -I_2R_2 + I_3R_3 &= 0 && \text{(loop } cfed) \end{aligned}$$

which can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ -R_1 & R_2 & 0 \\ 0 & -R_2 & R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

whose solution leads to the final result.

Section 2.5, p. 157

2. (b) and (c).

4. $\begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$.

6. (a) $\mathbf{x}^{(1)} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 0.06 \\ 0.24 \\ 0.70 \end{bmatrix}$, $\mathbf{x}^{(3)} = \begin{bmatrix} 0.048 \\ 0.282 \\ 0.670 \end{bmatrix}$, $\mathbf{x}^{(4)} = \begin{bmatrix} 0.056 \\ 0.286 \\ 0.658 \end{bmatrix}$.

(b) Let $*$ stand for any positive matrix entry. Then

$$T^2 = \begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix} \cdot \begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix},$$

$$T^3 = T^2 \cdot T = \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \cdot \begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} > 0,$$

hence T is regular; $\mathbf{u} = \begin{bmatrix} 0.057 \\ 0.283 \\ 0.660 \end{bmatrix}$.

8. In general, each matrix is regular, so T^n converges to a state of equilibrium. Specifically, $T^n \to$ a matrix all of whose columns are $\mathbf{u}$, where

$$\text{(a) } \mathbf{u} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \text{(b) } \mathbf{u} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}, \quad \text{(c) } \mathbf{u} = \begin{bmatrix} \frac{9}{17} \\ \frac{4}{17} \\ \frac{4}{17} \end{bmatrix}, \quad \text{(d) } \mathbf{u} = \begin{bmatrix} 0.333 \\ 0.111 \\ 0.555 \end{bmatrix}.$$

10. $$\text{(a) } \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} \quad \text{(b) } \begin{bmatrix} \frac{1}{8} \\ \frac{7}{8} \end{bmatrix} \quad \text{(c) } \begin{bmatrix} \frac{4}{11} \\ \frac{4}{11} \\ \frac{3}{11} \end{bmatrix} \quad \text{(d) } \begin{bmatrix} \frac{1}{11} \\ \frac{4}{11} \\ \frac{6}{11} \end{bmatrix}.$$

12. (a) $T = \begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}$. (b) $T \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.39 \\ 0.61 \end{bmatrix}$; 39% will order a subscription.

14. red, 25%; pink, 50%; white, 25%.

T.1. No. If the sum of the entries of each column is 1, it does not follow that the sum of the entries in each column of A^T will also be 1.

ML.2. Enter the matrix T and initial state vector $\mathbf{x}^{(0)}$ into MATLAB.

T = [.5 .6 .4;.25 .3 .3;.25 .1 .3];

x0 = [.1 .3 .6];

State vector $\mathbf{x}^{(5)}$ is given by

x5 = T^5 * x0

```
x5 =
      0.5055
      0.2747
      0.2198
```

ML.3. The command **sum** operating on a matrix computes the sum of the entries in each column and displays these totals as a row vector. If the output from the **sum** command is a row of ones, then the matrix is a Markov matrix.

(a) **A = [2/3 1/3 1/2;1/3 1/3 1/4;0 1/3 1/4];sum(A)**

```
ans =
      1   1   1
```

Hence A is a Markov matrix.

(b) **A = [.5 .6 .7;.3 .2 .3;.1 .2 0];sum(A)**

```
ans =
      0.9000   1.0000   1.0000
```

A is not a Markov matrix.

(c) **A = [.66 .25 .125;.33 .25 .625;0 .5 .25];sum(A)**

```
ans =
      0.9900   1.0000   1.0000
```

A is not a Markov matrix.

Section 2.6, p. 165

2. $\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$. 4. $\begin{bmatrix} 15 \\ 3 \\ 14 \end{bmatrix}$.

6. C_1's income is 24; C_2's income is 25; C_3's income is 16.

8. Productive 10. Productive.

12. \$2.805 million of copper, \$2.125 million of transportation, \$4.158 million of electric power.

T.1. We must show that for an exchange matrix A and vector $\mathbf{p}$, $A\mathbf{p} \le \mathbf{p}$ implies $A\mathbf{p} = \mathbf{p}$. Let $A = [a_{ij}]$, $\mathbf{p} = [p_j]$, and $A\mathbf{p} = \mathbf{y} = [y_j]$. Then

$$\sum_{j=1}^{n} y_j = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} p_k = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{jk} \right) p_k = \sum_{k=1}^{n} p_k$$

since the sum of the entries in the kth column of A is 1.

Since $y_j \le p_j$ for $j = 1, \dots, n$ and $\sum y_j = \sum p_j$, the respective entries must be equal: $y_j = p_j$ for $j = 1, \dots, n$. Thus $A\mathbf{p} = \mathbf{p}$.

Section 2.7, p. 178

2. Final average: 14.75;
   Detail coefficients: 8.25, 4, −1.5
   Compressed data: 14.75, 8.25, 4, 0
   Wavelet y-coordinates: 27, 19, 6.5, 8.5.

4. Final average: −0.875
   Detail coefficients: 0.625, −2.25, 0.5, 3.5, −3.0, 3.0, 1.0
   Compressed data: −0.875, 0, −2.25, 0, 3.5, −3.0, 3.0, 0
   Wavelet y-coordinates: 0.375, −6.625, −1.625, 4.375, 2.125, −3.875, −0.875, −0.875.

6. Computing the reduced row echelon for of A_1 and A_2, we find that in each case we obtain I_4.

Supplementary Exercises, p. 179

2. Let $A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$. The rectangle has vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(0, 2)$. We must have

$$\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & r & s & t \\ 0 & r & s & t \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix},$$

so $r = 4$ and $4a = 4$. Hence, $a = 1$. Also,

$$\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix},$$

so $8 = s$ and $4a + 2b = s = 8$, which implies that $b = 2$.

Chapter 3

Determinants

Section 3.1, p. 192

2. (a) even. (b) odd. (c) even. (d) odd. (e) even. (f) even.

4. The number of inversions are: (a) 9, 6. (b) 8, 7. (c) 5, 6. (d) 2, 7.

6. (a) 2. (b) 24. (c) -30. (d) 2.

8. $|B| = 4$; $|C| = -8$; $|D| = -4$.

10. $\det(A) = \det(A^T) = 14$.

12. (a) $(\lambda - 1)(\lambda - 2)(\lambda - 3) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$. (b) $\lambda^3 - \lambda$.

14. (a) 1, 2, 3. (b) $-1, 0, 1$.

16. (a) -144. (b) -168. (c) 72.

18. (a) -120. (b) 29. (c) 9.

20. (a) -1 (b) -120. (c) -22.

22. (a) 16. (b) 256. (c) $-\frac{1}{4}$.

24. (a) 1. (b) 1. (c) 1.

26. (a) 1. (b) 1.

T.1. If j_i and j_{i+1} are interchanged, all inversions between numbers distinct from j_i and j_{i+1} remain unchanged, and all inversions between one of j_i, j_{i+1} and some other number also remain unchanged. If originally $j_i < j_{i+1}$, then after interchange there is one additional inversion due to $j_{i+1}j_i$. If originally $j_i > j_{i+1}$, then after interchange there is one fewer inversion.

Suppose $j_p$and j_q are separated by k intervening numbers. Then k interchanges of adjacent numbers will move j_p next to j_q. One interchange switches j_p and j_q. Finally, k interchanges of adjacent numbers takes j_q back to j_p's original position. The total number of interchanges is the odd number $2k + 1$.

T.2. Parallel to proof for the upper triangular case.

T.3. $cA = \left[ca_{ij}\right]$. By n applications of Theorem 3.5, the result follows.

T.4. If A is nonsingular, then $AA^{-1} = I_n$. Therefore $\det(A) \cdot \det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$. Thus $\det(A) \neq 0$ and

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

T.5. $\det(AB) = \det(A)\det(B)$. Thus if $\det(AB) = 0$, then $\det A \cdot \det B = 0$, and either $\det A = 0$ or $\det B = 0$.

T.6. $\det(AB) = \det(A) \cdot \det(B) = \det(B) \cdot \det(A) = \det(BA)$.

T.7. In the summation

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

for the definition of $\det(A)$ there is exactly one nonzero term. Thus $\det(A) \neq 0$.

T.8. $\det(A)\det(B) = \det(AB) = \det(I_n) = 1$. Thus $\det(A) \neq 0$ and $\det(B) \neq 0$.

T.9. (a) $[\det(A)]^2 = \det(A)\det(A) = \det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$.

(b) $[\det(A)]^2 = \det(A)\det(A) = \det(A)\det(A^T) = \det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$.

T.10. $\det(A^2) = [\det(A)]^2 = \det(A)$, so $\det(A)$ is a nonzero root of the equation $x^2 - x = 0$.

T.11. $\det(A^T B^T) = \det(A^T)\det(B^T) = \det(A)\det(B^T) = \det(A^T)\det(B)$.

T.12. $$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 - a^2 & b - a & 0 \\ c^2 - a^2 & c - a & 0 \end{vmatrix} = \begin{vmatrix} (b-a)(b+a) & b-a \\ (c-a)(c+a) & c-a \end{vmatrix}$$
$$= (b-a)(c-a) \begin{vmatrix} b+a & 1 \\ c+a & 1 \end{vmatrix} = (b-a)(c-a)(b-c).$$

T.13. If A is nonsingular, by Corollary 3.2, $\det(A) \neq 0$ and $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$. Conversely, if $a_{ii} \neq 0$ for $i = 1, \ldots, n$, then clearly A is row equivalent to I_n, and thus is nonsingular.

T.14. $\det(AB) = \det(A)\det(B) = 0 \cdot \det(B) = 0$.

T.15. If $\det(A) \neq 0$, then since

$$0 = \det(O) = \det(A^n) = \det(A)\det(A^{n-1}),$$

by Exercise T.5 above, $\det(A^{n-1}) = 0$. Working downward, $\det(A^{n-2}) = 0, \ldots, \det(A^2) = 0$, $\det(A) = 0$, which is a contradiction.

T.16. $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = \det(A)$, which implies $\det(A) = 0$.

T.17. Follows immediately from Theorem 3.7.

T.18. When all the entries on its main diagonal are nonzero.

T.19. Ten have determinant 0 and six have determinant 1.

ML.1. There are many sequences of row operations that can be used. Here we record the value of the determinant so you may check your result.

(a) $\det(A) = -18$. (b) $\det(A) = 5$.

ML.2. There are many sequences of row operations that can be used. Here we record the value of the determinant so you may check your result.

(a) $\det(A) = -9$. (b) $\det(A) = 5$.

ML.3. (a)
```
A = [1  -1  1;1  1 -1;-1  1  1];
det(A)
ans =
      4
```
(b)
```
A = [1  2  3  4;2  3  4  5;3  4  5  6;4  5  6  7];
det(A)
ans =
      0
```

ML.4. (a)
```
A = [2  3  0;4  1  0;0  0  5];
det(5 * eye(size(A)) - A)
ans =
      0
```
(b)
```
A = [1  1;5  2];
det(3 * eye(size(A)) - A)^2
ans =
      9
```
(c)
```
A = [1  1  0;0  1  0;1  0  1];
det(inverse(A) * A)
ans =
      1
```

ML.5.
```
A = [5  2;-1  2];
t = 1;
det(t * eye(size(A)) - A)
ans =
      6
t = 2;
det(t * eye(size(A)) - A)
ans =
      2
t = 3;
det(t * eye(size(A)) - A)
ans =
      0
```

Section 3.2, p. 207

2. $A_{21} = 0$, $A_{22} = 0$, $A_{23} = 0$, $A_{24} = 13$, $A_{13} = -9$, $A_{23} = 0$, $A_{33} = 3$, $A_{43} = -2$.

4. (a) 9. (b) 13. (c) -26.

6. (a) -135. (b) -20. (c) -20.

8. (a) $\begin{bmatrix} 2 & -7 & -6 \\ 1 & -7 & -3 \\ -4 & 7 & 5 \end{bmatrix}$. (b) -7.

10. (a) $\begin{bmatrix} \frac{2}{9} & -\frac{1}{9} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$. (b) $\begin{bmatrix} \frac{3}{14} & -\frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{5}{7} & -\frac{4}{7} \\ -\frac{1}{14} & \frac{1}{7} & \frac{2}{7} \end{bmatrix}$. (c) Singular.

12. (a) $\begin{bmatrix} 1 & 0 & -1 \\ -2 & \frac{1}{2} & \frac{5}{2} \\ -1 & 0 & 2 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{5}{3} \end{bmatrix}$. (c) $\begin{bmatrix} -\frac{1}{21} & -\frac{2}{21} & \frac{8}{21} \\ \frac{4}{21} & -\frac{5}{42} & -\frac{1}{42} \\ \frac{7}{42} & \frac{7}{84} & -\frac{7}{84} \end{bmatrix}$.

14. (a), (b) and (d) are nonsingular.

16. (a) 0, 5. (b) -1, 0, 1.

18. (a) Has nontrivial solutions. (b) Has only the trivial solution.

20. $x = -2$, $y = 0$, $z = 1$.

22. $x = \frac{22}{5}$, $y = -\frac{26}{5}$, $z = \frac{12}{5}$.

24. (a) is nonsingular.

T.1. Let A be upper triangular. Then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \cdots & \\ 0 & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ & \ddots & \\ 0 & \cdots & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}.$$

T.2. (a) $\det(A) = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21})$

$= -a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32}.$

T.3. The i, j entry of adj A is $A_{ji} = (-1)^{j+i}\det(M_{ji})$, where M_{ji} is the submatrix of A obtained by deleting from A the jth row and ith column. Since A is symmetric, that submatrix is the transpose of M_{ij}. Thus

$$A_{ji} = (-1)^{j+i}\det(M_{ji}) = (-1)^{i+j}\det(M_{ij}) = j,\ i \text{ entry of adj } A.$$

Thus adj A is symmetric.

T.4. The adjoint matrix is upper triangular if A is upper triangular, since $A_{ij} = 0$ if $i > j$.

T.5. If $\det(A) = ad - bc \neq 0$, then by Corollary 3.3,

$$A^{-1} = \frac{1}{\det(A)}(\text{adj } A) = \frac{1}{ad - bc} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}.$$