Chapter 6

Real Vector Spaces

Section 6.1, p. 278

2. Closed under $\oplus$; closed under $\odot$.

4. Closed under $\oplus$; closed under $\odot$.

8. Properties (α), (a), (b), (c), (d) follow as for R^3.

   Regarding (β): (cx, y, z) is a triple of real numbers, so it lies in V.

   Regarding (g): $c \odot (d \odot (x, y, z)) = c \odot (dx, y, z) = (cdx, y, z) = (cd) \odot (x, y, z)$.

   Regarding (h): $1 \odot (x, y, z) = (1 \cdot x, y, z) = (x, y, z)$.

10. P is a vector space. Let $p(t)$ be a polynomial of degree n and $q(t)$ a polynomial of degree m, and $r = \max(n, m)$. Then inside the vector space P_r, addition of $p(t)$ and $q(t)$ is defined and multiplication of $p(t)$ by a scalar is defined, and these operations satisfy the vector space axioms. P_r is contained in P. Thus the additive inverse of $p(t)$, zero polynomial, etc. all lie in P. Hence P is a vector space.

12. Not a vector space; (e), (f), and (h) do not hold.

14. Vector space.

16. Not a vector space; (h) does not hold.

18. No. For example, (a) fails since $2\mathbf{u} - \mathbf{v} \neq 2\mathbf{v} - \mathbf{u}$.

20. (a) Infinitely many.

    (b) The only vector space having a finite number of vectors is $\{\mathbf{0}\}$.

T.1. $c\mathbf{u} = c(\mathbf{u} + \mathbf{0}) = c\mathbf{u} + c\mathbf{0} = c\mathbf{0} + c\mathbf{u}$ by Definition 1(c), (e) and (a). Add the negative of $c\mathbf{u}$ to both sides of this equation to get $\mathbf{0} = c\mathbf{0} + c\mathbf{u} + (-c\mathbf{u}) = c\mathbf{0} + \mathbf{0} = c\mathbf{0}$.

T.2. $-(-\mathbf{u})$ is that unique vector which when added to $-\mathbf{u}$ gives $\mathbf{0}$. But $\mathbf{u}$ added to $-\mathbf{u}$ gives $\mathbf{0}$. Thus $-(-\mathbf{u}) = \mathbf{u}$.

T.3. (cancellation): If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then

$$\begin{aligned}(-\mathbf{u}) + (\mathbf{u} + \mathbf{v}) &= (-\mathbf{u}) + (\mathbf{u} + \mathbf{w})\\ (-\mathbf{u} + \mathbf{u}) + \mathbf{v} &= (-\mathbf{u} + \mathbf{u}) + \mathbf{w}\\ \mathbf{0} + \mathbf{v} &= \mathbf{0} + \mathbf{w}\\ \mathbf{v} &= \mathbf{w}\end{aligned}$$

T.4. If $\mathbf{u} \neq \mathbf{0}$ and $a\mathbf{u} = b\mathbf{u}$, then $(a - b)\mathbf{u} = a\mathbf{u} - b\mathbf{u} = \mathbf{0}$. By Theorem 6.1(c), $a - b = 0$, $a = b$.

T.5. Let $\mathbf{0}_1$ and $\mathbf{0}_2$ be zero vectors. Then $\mathbf{0}_1 \oplus \mathbf{0}_2 = \mathbf{0}_1$ and $\mathbf{0}_1 \oplus \mathbf{0}_2 = \mathbf{0}_2$. So $\mathbf{0}_1 = \mathbf{0}_2$.

T.6. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be negatives of $\mathbf{u}$. Then $\mathbf{u} \oplus \mathbf{u}_1 = \mathbf{0}$ and $\mathbf{u} \oplus \mathbf{u}_2 = \mathbf{0}$. So $\mathbf{u} \oplus \mathbf{u}_1 = \mathbf{u} \oplus \mathbf{u}_2$. Then

$$\begin{aligned} \mathbf{u}_1 \oplus (\mathbf{u} \oplus \mathbf{u}_1) &= \mathbf{u}_1 \oplus (\mathbf{u} \oplus \mathbf{u}_2) \\ (\mathbf{u}_1 \oplus \mathbf{u}) \oplus \mathbf{u}_1 &= (\mathbf{u}_1 \oplus \mathbf{u}) \oplus \mathbf{u}_2 \\ \mathbf{0} \oplus \mathbf{u}_1 &= \mathbf{0} \oplus \mathbf{u}_2 \\ \mathbf{u}_1 &= \mathbf{u}_2. \end{aligned}$$

T.7. The sum of any pair of vectors from B^n is, by virtue of entry-by-entry binary addition, a vector in B^n. Thus B^n is closed.

T.8. For $\mathbf{v}$ any vector if B^n, we have $0\mathbf{v} = \mathbf{0}$ and $1\mathbf{v} = \mathbf{v}$. Both $\mathbf{0}$ and $\mathbf{v}$ are in B^n, so B^n is closed under scalar multiplication.

T.9. Let $\mathbf{v} = (b_1, b_2, \ldots, b_n)$ be in B^n. Then $1\mathbf{v} = (1b_1, 1b_2, \ldots, 1b_n) = \mathbf{v}$.

ML.2 **p = [2 5 1 − 2],q = [1 0 3 5]**

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p =
   2   5   1  -2

q =
   1   0   3   5
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(a) **p + q**

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ans =
      3   5   4   3
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which is $3t^3 + 5t^2 + 4t + 3$.

(b) **5 * p**

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ans =
     10   25   5   -10
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which is $10t^3 + 25t^2 + 5t - 10$.

(c) **3 * p − 4 * q**

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ans =
      2   15   -9   -26
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which is $2t^3 + 15t^2 - 9t - 26$.

Section 6.2, p. 287

2. Yes.

4. No.

6. (a) and (b).

8. (a).

10. (a) and (b).

12. Since P_n is a subset of P and it is a vector space with respect to the operations in P, it is a subspace of P.

Since the system is consistent, $p(t)$ is in span S.

Section 6.3, p. 301

2. (c) and (d).

4. (a) and (c).

6. $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

8. No.

10. (b) $(2,6,-5) = (4,2,1) - 2(1,-2,3)$.

    (c) $(3,6,6) = 2(1,1,0) + (0,2,3) + (1,2,3)$.

12. (c) $3t + 1 = 3(2t^2 + t + 1) - 2(3t^2 + 1)$.

    (d) $5t^2 - 5t - 6 = 2(t^2 - 4) + (3t^2 - 5t + 2)$.

14. Only (d) is linearly dependent: $\cos 2t = \cos^2 t - \sin^2 t$.

16. $\lambda \neq \pm 2$.

18. Yes.

20. Yes.

22. $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3$.

T.1. If $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n = (c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0) = \mathbf{0}$ in R^n, then $c_1 = c_2 = \cdots = c_n = 0$.

T.2. (a) Since S_1 is linearly dependent, there are vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in S_1 and constants $c_1, c_2, \ldots, c_k$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Those $\mathbf{v}_i$'s also lie in S_2, hence S_2 is linearly dependent.

(b) Suppose S_1 were linearly dependent, then by part (a), S_2 would be linearly dependent. Contradiction.

T.3. Assume that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent. Then there are constants c_i, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Let c_j be a nonzero coefficient. Then, solving the equation for $\mathbf{v}_j$, we find that

$$\mathbf{v}_j = -\frac{c_1}{c_j}\mathbf{v}_1 - \frac{c_2}{c_j}\mathbf{v}_2 - \cdots - \frac{c_{j-1}}{c_j}\mathbf{v}_{j-1} - \frac{c_{j+1}}{c_j}\mathbf{v}_{j+1} - \cdots - \frac{c_k}{c_j}\mathbf{v}_k.$$

Conversely, if

$$\mathbf{v}_j = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_{j-1}\mathbf{v}_{j-1} + d_{j+1}\mathbf{v}_{j+1} + \cdots + d_k\mathbf{v}_k$$

for some coefficients d_i, then

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + (-1)\mathbf{v}_j + \cdots + d_k\mathbf{v}_k = \mathbf{0}$$

and the set S is linearly dependent.

T.4. Suppose

$$\begin{aligned} c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 &= c_1(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + c_2(\mathbf{v}_2 + \mathbf{v}_3) + c_3\mathbf{v}_3 \\ &= c_1\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 + (c_1 + c_2 + c_3)\mathbf{v}_3 = \mathbf{0}. \end{aligned}$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, $c_1 = 0$, $c_1 + c_2 = 0$ (and hence $c_2 = 0$), and $c_1 + c_2 + c_3 = 0$ (and hence $c_3 = 0$). Thus the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

T.5. Form the linear combination

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$

which gives

$$c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) = (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3 = \mathbf{0}.$$

Since S is linearly independent, we have

$$\begin{aligned} c_1 + c_2 \qquad &= 0 \\ c_1 \qquad + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

a linear system whose augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

The reduced row echelon form is

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

thus $c_1 = c_2 = c_3 = 0$ which implies that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

T.6. Form the linear combination

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$

which gives

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = (c_1 + c_2 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Since S is linearly dependent, this last equation is satisfied with $c_1 + c_2 + c_3$, $c_2 + c_3$, and c_3 not all being zero. This implies that c_1, c_2, and c_3 are not all zero. Hence, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly dependent.

T.7. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Then one of the $\mathbf{v}_j$'s is a linear combination of the preceding vectors in the list. It must be $\mathbf{v}_3$ since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Thus $\mathbf{v}_3$ belongs to span $\{\mathbf{v}_1, \mathbf{v}_2\}$. Contradiction.

T.8. Let $\mathbf{a}_1, \ldots, \mathbf{a}_r$ be the nonzero rows of the reduced row echelon form matrix A, and suppose

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_r\mathbf{a}_r = \mathbf{0}. \tag{6.2}$$

For each j, $1 \le j \le r$, $\mathbf{a}_j$ is the only row with a nonzero entry in the column which holds the leading entry of that row. Thus, in the summation (6.2), c_j must be zero. Hence (6.2) is the trivial dependence relation, and the $\mathbf{a}_i$ are linearly independent.

Section 6.4, p. 314

2. (c).

4. (d).

6. If

$$c_1\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}+c_2\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}+c_3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+c_4\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix},$$

then

$$\begin{bmatrix}c_1+c_3 & c_1+c_4\\c_2+c_4 & c_2+c_3+c_4\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

The first three entries imply that $c_3=-c_1=c_4=-c_2$. The fourth entry gives $c_2-c_2-c_2=-c_2=0$. Thus $c_i=0$ for $i=1,2,3,4$. Hence the set of four matrices is linearly independent. By Theorem 6.9, it is a basis.

8. (b); $(2,1,3)=1(1,1,2)+2(2,2,0)-1(3,4,-1)$.

10. (a) forms a basis: $5t^2-3t+8=-3(t^2+t)+0t^2+8(t^2+1)$.

12. Possible answer: $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$; $\dim W=3$.

14. Possible answer: $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$.

16. Possible answer: $\{\cos^2 t,\sin^2 t\}$; $\dim W=2$.

18. (a) $\{(0,1,0),(0,0,1)\}$. (b) $\{(1,0,1,0),(0,1,-1,-1)\}$. (c) $\{(1,1,0),(-5,0,1)\}$.

20. (a) 3. (b) 2.

22. $\{t^3+t^2,t+1\}$.

24. (a) 2. (b) 3. (c) 3. (d) 3.

26. 2.

28. (a) Possible answer: $\{(1,0,2),(1,0,0),(0,1,0)\}$.
    (b) Possible answer: $\{(1,0,2),(0,1,3),(1,0,0)\}$.

30. For $a\neq -1,0,1$.

32. $S=\left\{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\right\}$.

34. The set of all polynomials of the form $at^3+bt^2+(b-a)$, where a and b are any real numbers.

36. Possible answer: $\{(-1,1,0),(3,0,1)\}$.

38. Yes.

40. No.

T.1. Since the largest number of vectors in any linearly independent set is m, $\dim V=m$. The result follows from Theorem 6.9.