Name: ______________________ ID#: ______________

Solutions to Final Exam

Math 20
Introduction to Linear Algebra
and Multivariable Calculus

21 May 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> *—Handbook for Students*

1. (20 Points) For this and the next three problems, let

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

(a) Find the reduced row echelon form of A. ***Label each operation to receive partial credit in case of arithmetic mistakes.***

Solution. We have

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{matrix} \leftarrow (-1) \cdot \text{row } 1 \\ \leftarrow \text{row } 2 + (-2) \cdot \text{row } 1 \\ \leftarrow \text{row } 3 + (-1) \cdot \text{row } 1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow \tfrac{1}{2} \text{row } 2 \\ \leftarrow \text{row } 3 + (-1) \cdot \text{row } 2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 + (-1) \cdot \text{row } 2 \\ \\ \\ \end{matrix}$$

□

(b) Find each of the following numbers. Explain your answers.

_____ *(i)* rank A

Solution. The rank is the dimension of the column space, or the number of columns of the RREF which are standard basis vectors, or the number of leading entries in the RREF. This number is apparently 2. □

_____ *(ii)* null A

Solution. The rank plus the nullity is the number of columns in a matrix. In our case, this is 1. □

_____ *(iii)* det A

Solution. Since the nullity is positive, the matrix is not invertible, so its determinant is zero. □

_____ / 20

2. (15 Points) *Remember,*

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

Let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find the parametric form of the general solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$.

Solution. We form the augmented matrix and find its RREF.

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -2 & 0 & 2 & 2 \\ -1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore we can read off the general solution: x_3 is free, $x_2 = 0$, and $x_1 = -1 + x_3$. Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

□

______ / 15

3. (15 Points) *Continuing to let*

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix},$$

let

$$\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Find the parametric form of the general least-squares *solution to the system of linear equations* $A\mathbf{x} = \mathbf{c}$.

Solution. A least-squares solution to $A\mathbf{x} = \mathbf{c}$ is a solution to $A^T A\mathbf{x} = A^T\mathbf{c}$. We have

$$A^T A = \begin{bmatrix} 6 & 0 & -6 \\ 0 & 2 & 0 \\ -6 & 0 & 6 \end{bmatrix}; \qquad A^T\mathbf{c} = \begin{bmatrix} -7 \\ 1 \\ 7 \end{bmatrix}.$$

Using Gaussian Elimination,

$$\begin{bmatrix} A^T A & A^T\mathbf{c} \end{bmatrix} = \begin{bmatrix} 6 & 0 & -6 & -7 \\ 0 & 2 & 0 & 1 \\ -6 & 0 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -\frac{7}{6} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution is x_3 free, $x_2 = -\frac{1}{2}$, $x_1 = -\frac{7}{6} + x_3$, so

$$\mathbf{x} = \begin{bmatrix} -\frac{7}{6} + x_3 \\ -\frac{1}{2} \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{6} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

□

______ / 15

4. (20 Points) *Don't forget,*

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}.$$

Solution. First we have to find the eigenvalues of A. The are the roots λ of the polynomial

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & -1 & 1 \\ -2 & -\lambda & 2 \\ -1 & 1 & 1-\lambda \end{vmatrix} \\ &= (-1)\begin{vmatrix} -1 & 1 \\ -\lambda & 2 \end{vmatrix} - \begin{vmatrix} -1-\lambda & 1 \\ -2 & 2 \end{vmatrix} + (1-\lambda)\begin{vmatrix} -1-\lambda & -1 \\ -2 & -\lambda \end{vmatrix} \\ &= (-1)\,[-2+\lambda] - [-2(\lambda+1)+2] + (1-\lambda)\,[(\lambda+1)\lambda - 2] \\ &= 4\lambda - \lambda^3 = -\lambda(\lambda^2 - 4). \end{aligned}$$

So the eigenvalues of A are 0 and ± 2. To find eigenvectors, we look for a basis to the null space of each $A - \lambda I$. $A - 0I = A$ has already be reduced in Problem 1.

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From there, we have the general solution to $A\mathbf{x} = \mathbf{0}$ as x_3 free, $x_2 = 0$, $x_1 = x_3$. Therefore a vector spanning the null space of A is

$$\mathbf{v_0} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For the eigenvalue 2,

$$\begin{aligned} A - 2I = \begin{bmatrix} -3 & -1 & 1 \\ -2 & -2 & 2 \\ -1 & 1 & -1 \end{bmatrix} &\sim \begin{bmatrix} -1 & 1 & -1 \\ -2 & -2 & 2 \\ -3 & -1 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{row } 3 \\ \\ \leftarrow \text{row } 1 \end{matrix} \\ &\sim \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{row } 2 + 2 \cdot \text{row } 1 \\ \leftarrow \text{row } 3 + 3 \cdot \text{row } 1 \end{matrix} \\ &\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow (-1) \cdot \text{row } 1 \\ \leftarrow (\frac{1}{2}) \cdot \text{row } 2 \\ \\ \end{matrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 + \text{row } 2 \\ \\ \\ \end{matrix} \end{aligned}$$

The general solution to $A\mathbf{x} = 2\mathbf{x}$ is therefore x_3 free, $x_2 = x_3$, $x_1 = 0$. So a vector spanning this eigenspace is

$$\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For the last eigenvalue, we have

$$\begin{aligned} A + 2I = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{row } 2 + 2 \cdot \text{row } 1 \\ \leftarrow \text{row } 3 + \text{row } 1 \end{array} \\ &\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \leftarrow \frac{1}{4} \cdot \text{row } 2 \\ \leftarrow \text{row } 3 + (-1) \cdot \text{row } 2 \end{array} \\ &\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow \text{row } 1 + (-1) \cdot \text{row } 2 \\ \\ \\ \end{array} \end{aligned}$$

So the general solution to $A\mathbf{x} = -2\mathbf{x}$ is $x_3 = 0$, x_2 free, $x_1 = x_2$. Therefore a vector spanning this eigenspace is

$$\mathbf{v_{-2}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Assembling this all into matrices, we set

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \tag{1}$$

$$P = \begin{bmatrix} \mathbf{v_0} & \mathbf{v_2} & \mathbf{v_{-2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{2}$$

You may check for yourself that $AP = PD$, which means $A = PDP^{-1}$. □

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5. (15 Points) *Now let*

$$B = \begin{bmatrix} -1 & -1 & 0 \\ -2 & 2 & 4 \\ -1 & 1 & 2 \end{bmatrix}.$$

Find an orthonormal basis for $\operatorname{Col} B$.

Solution. First we have to find any basis for $\operatorname{Col} B$. Since

$$B \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first columns of the RREF span its column space, the first two columns of B span $\operatorname{Col} B$. This basis can be orthogonalized through the Gram-Schmidt process:

$$\mathbf{u}_1 = \mathbf{b}_1 = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix};$$

$$\mathbf{u}_2 = \mathbf{b}_2 - \frac{\mathbf{b}_2 \cdot \mathbf{u}_1}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{1}{3}\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}.$$

□

We can normalize the basis by dividing each vector by its length, and scaling $\mathbf{u}_1$ *by* -1*:*

$$\mathbf{w}_1 = \frac{-\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix};$$

$$\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{30}}\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix};$$

______ / 15

6. (15 Points) *Let T be the linear transformation which reflects the plane through the line $y = x$. Give the eigenvalues and eigenvectors of this transformation.*

Hint. You could find a matrix for this linear transformation and then diagonalize it; however, it might be easier to think about what this linear transformation does and what eigenvectors/eigenvalues are.

Solution. Vectors parallel to the line $y = x$ are not going to be changed by T. Therefore, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1. On the other hand, vectors orthogonal to the line $y = x$ will be exactly flipped, so $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue -1. □

______ / 15

7. (15 Points) *Let C be a matrix whose columns are linearly independent, so that $C^T C$ is invertible. Let*

$$P = C(C^T C)^{-1} C^T$$

Show:

(i) $P^2 = P$.

Solution. We write down P^2 and follow our nose:

$$P^2 = C(C^T C)^{-1} C^T C(C^T C)^{-1} C^T = C(C^T C)^{-1} C^T = P.$$

□

(ii) $P\mathbf{v} = \mathbf{v}$ *for all* $\mathbf{v} \in \operatorname{Col} C$.

Hint. If $\mathbf{v} \in \operatorname{Col} C$, there exists $\mathbf{w}$ such that $\mathbf{v} = C\mathbf{w}$.

Solution. Starting with the hint, we have

$$P\mathbf{v} = C(C^T C)^{-1} C^T (C\mathbf{w}) = C\mathbf{w} = \mathbf{v}.$$

□

(iii) $P\mathbf{v} = \mathbf{0}$ *for all* $\mathbf{v} \in (\operatorname{Col} C)^\perp$.

Solution. Remember that $(\operatorname{Col} C)^\perp = \operatorname{Null} C^T$, so if $\mathbf{v} \in (\operatorname{Col} C)^\perp$, then

$$P\mathbf{v} = C(C^T C)^{-1} C^T \mathbf{v} = C(C^T C)^{-1} \mathbf{0} = \mathbf{0}.$$

□

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8. (10 Points) *Show that the function* $u(x, y) = e^{-13t} \sin 2x \cos 3y$ *satisfies the heat equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}.$$

Solution. We only have to take partial derivatives and compare:

$$\frac{\partial u}{\partial x} = 2e^{-13t} \cos 2x \cos 3y;$$
$$\frac{\partial^2 u}{\partial x^2} = -4e^{-13t} \sin 2x \cos 3y;$$
$$\frac{\partial u}{\partial y} = -3e^{-13t} \cos 2x \sin 3y;$$
$$\frac{\partial^2 u}{\partial y^2} = -9e^{-13t} \sin 2x \cos 3y;$$
$$\frac{\partial u}{\partial t} = -13e^{-13t} \sin 2x \cos 3y.$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$ can be easily verified. □

______ / 10

9. (20 Points) *Let* $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$*. Find and describe all critical points of* f*.*

Solution. We have

$$\frac{\partial f}{\partial x} = 6xy - 6x; \qquad \frac{\partial f}{\partial y} = 3x^2 + 3y^2 - 6y.$$

To see when these two equations are simultaneously zero, we first solve

$$0 = 6xy - 6x = 6x(y-1),$$

which is zero exactly when $x = 0$ or $y = 1$. In the first case, substituting $x = 0$ into $\dfrac{\partial f}{\partial y} = 0$ gives

$$0 = 3y^2 - 6y = 3y(y-2)$$

which means $y = 0$ or $y = 2$. Thus $(0,0)$ and $(0,2)$ are two critical points for f. In the second case, substituting $y = 1$ into $\dfrac{\partial f}{\partial y} = 0$ gives

$$0 = 3x^2 - 3 = 3(x+1)(x-1),$$

so $x = \pm 1$. Therefore $(1,1)$ and $(-1,1)$ are critical points, too.

The nature of the critical points can be determined by the Hessian matrix

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6y-6 & 6x \\ 6x & \\ 6y-6 & \end{bmatrix} . = 6 \begin{bmatrix} y-1 & x \\ x & \\ y-1 & \end{bmatrix}$$

We have

$$H(0,0) = 6\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which has positive determinant and negative trace. Hence $(0,0)$ is a local maximum.

$$H(0,2) = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which has positive determinant and positive trace. Hence $(0,2)$ is a local minimum.

$$H(1,1) = 6\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which has negative determinant. So $(1,1)$ is a saddle point. Finally,

$$H(-1,1) = 6\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

which also has negative determinant. So $(-1,1)$ is a saddle point, too.

Some graphs of f might clarify:

9

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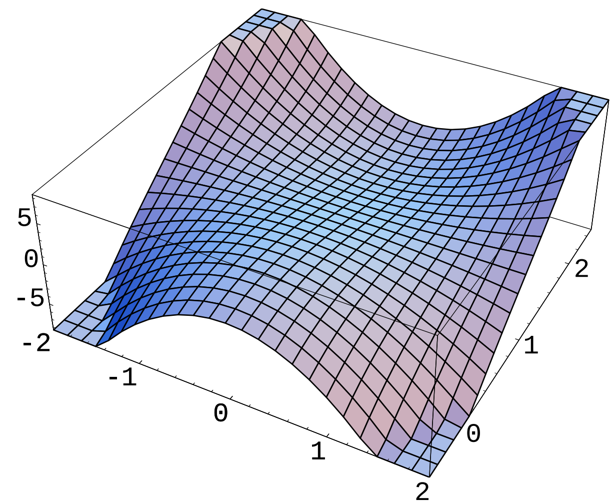

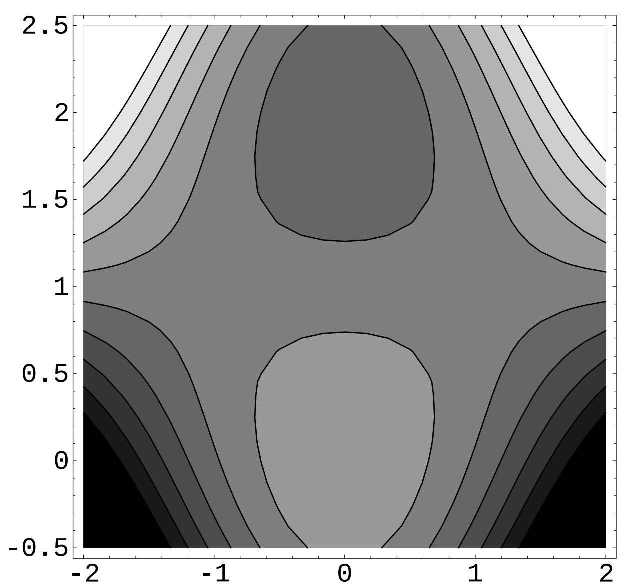

□

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10. (20 Points) *Let a, b, and c be positive constants. Find (x, y, z) such that $x + y + z = 1$ and $x^a y^b z^c$ is maximized.*

Solution. Let $f(x, y, z) = x^a y^b z^c$ and $g(x, y, z) = x+y+z$. We want to maximize f subject to $g = 1$. Then there should exist λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z).$$

By taking the derivatives, this vector equation becomes the system

$$\begin{aligned} a x^{a-1} y^b z^c &= \lambda; \\ b x^a y^{b-1} z^c &= \lambda; \\ c x^a y^b z^{c-1} &= \lambda. \end{aligned}$$

Dividing the second of these by the first gives

$$1 = \frac{b x^a y^{b-1} z^c}{a x^{a-1} y^b z^c} = \frac{bx}{ay},$$

so $y = \frac{b}{a} x$. Dividing the third by the first gives

$$1 = \frac{c x^a y^b z^{c-1}}{a x^{a-1} y^b z^c} = \frac{cx}{az},$$

so $z = \frac{c}{a} x$. Substituting both of these into the equation of constraint gives

$$1 = \left(1 + \frac{b}{a} + \frac{c}{a}\right) x = \frac{a+b+c}{a} x.$$

This gives us

$$\begin{aligned} x &= \frac{a}{a+b+c}; \\ y &= \frac{b}{a+b+c}; \\ z &= \frac{c}{a+b+c}. \end{aligned}$$

□

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11. (15 Points) *Label the following statements as true or false. Justify your answers. If true, cite appropriate facts or theorems. If false, give a* counterexample, *i.e., an example in which the hypothesis (the "if" part of the proposition) is satsified, but not the conclusion ("then").*

______ *(i) If B is a matrix with characteristic polynomial $\lambda^2 - 3\lambda + 2$, then B is diagonalizable.*

Solution. This polynomial factors as $(\lambda - 2)(\lambda - 1)$. Therefore B has two distinct eigenvalues and therefore two linearly independent eigenvectors. So B is diagonalizable; the statement is *true.* □

______ *(ii) If three vectors are linearly dependent, one must be a multiple of another.*

Solution. The statement is *false.* Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linearly independent but none of the vectors is a multiple of another. □

______ *(iii) If u is a differentiable function of several variables, then ∇u points in the direction of greatest change.*

Solution. This statement is *true.* □

______ *(iv) Two planes in $\mathbb{R}^3$ intersect in a line.*

Solution. This statement is *false.* Consider the parallel planes $z = 0$ and $z = 1$. These do not intersect. □

______ *(v) If $\mathcal{S}$ is a finite set of vectors in $\mathbb{R}^n$, then $(\mathcal{S}^\perp)^\perp = \mathcal{S}$.*

Solution. If S is any set, then $S^\perp$ is a *subspace* of $\mathbb{R}^n$. If $\mathcal{S} = \{\mathbf{e}_1\}$, then $(\mathcal{S}^\perp)^\perp$ is the subspace *spanned by* $\mathbf{e}_1$, not $\mathbf{e}_1$ alone. The statement is *false.* □

______ / 15

12. (20 Points) *State as many conditions as you can think of for a square matrix to be invertible (2 points each, maximum of 10).*

Solution. Here are many equivalent conditions as outlined in Lay.

(a) A is invertible.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

(e) The columns of A are linearly independent.

(f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I$.

(k) There is an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

(m) The columns of A form a basis of $\mathbb{R}^n$.

(n) $\operatorname{Col} A = \mathbb{R}^n$.

(o) $\dim \operatorname{Col} A = n$.

(p) $\operatorname{rank} A = n$.

(q) $\operatorname{Null} A = \{\mathbf{0}\}$.

(r) $\operatorname{null} A = 0$.

(s) 0 is not an eigenvalue of A.

(t) $\det A \neq 0$.

(u) $(\operatorname{Col} A)^\perp = \{\mathbf{0}\}$.

(v) $(\operatorname{Null} A)^\perp = \mathbb{R}^n$.

(w) $\operatorname{Row} A = \mathbb{R}^n$.

□

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