Name: ______________________________ ID#: ____________________

Final Exam

Math 20
Introduction to Linear Algebra
and Multivariable Calculus

21 May 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

> *Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*
>
> *—Handbook for Students*

Problem Number	Possible Points	Points Earned
1	20	
2	15	
3	15	
4	20	
5	15	
6	15	
7	15	
8	10	
9	20	
10	20	
11	15	
12	20	
Total	200	

Ave Cæsar
Morituri Te Salutant
—Gladiator Salute

1. (20 Points) *For this and the next three problems, let*

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

(a) Find the reduced row echelon form of A. **Label each operation to receive partial credit in case of arithmetic mistakes.**

(b) Find each of the following numbers. Explain your answers.

_______ (i) $\operatorname{rank} A$

_______ (ii) $\operatorname{null} A$

_______ (iii) $\det A$

_______ / 20

2. (15 Points) Remember,

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

Let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find the parametric form of the general solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$.

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3. (15 Points) Continuing to let

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix},$$

let

$$\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Find the parametric form of the general *least-squares* solution to the system of linear equations $A\mathbf{x} = \mathbf{c}$.

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4. (20 Points) Don't forget,

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}.$$

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5. (15 Points) Now let

$$B = \begin{bmatrix} -1 & -1 & 0 \\ -2 & 2 & 4 \\ -1 & 1 & 2 \end{bmatrix}.$$

Find an orthonormal basis for $\operatorname{Col} B$.

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6. (15 Points) Let T be the linear transformation which reflects the plane through the line $y = x$. Give the eigenvalues and eigenvectors of this transformation.

Hint. You could find a matrix for this linear transformation and then diagonalize it; however, it might be easier to think about what this linear transformation does and what eigenvectors/eigenvalues are.

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7. (15 Points) Let C be a matrix whose columns are linearly independent, so that $C^T C$ is invertible. Let

$$P = C(C^T C)^{-1} C^T$$

Show:

(i) $P^2 = P$.

(ii) $P\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \operatorname{Col} C$.

Hint. If $\mathbf{v} \in \operatorname{Col} C$, there exists $\mathbf{w}$ such that $\mathbf{v} = C\mathbf{w}$.

(iii) $P\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in (\operatorname{Col} C)^{\perp}$.

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8. (10 Points) Show that the function $u(x,y) = e^{-13t} \sin 2x \cos 3y$ satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}.$$

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9. (20 Points) Let $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$. Find and describe all critical points of f.

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10. (20 Points) Let a, b, and c be positive constants. Find (x, y, z) such that $x + y + z = 1$ and $x^a y^b z^c$ is maximized.

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11. (15 Points) Label the following statements as true or false. Justify your answers. If true, cite appropriate facts or theorems. If false, give a *counterexample*, i.e., an example in which the hypothesis (the "if" part of the proposition) is satsified, but not the conclusion ("then").

_____ (i) If B is a matrix with characteristic polynomial $\lambda^2 - 3\lambda + 2$, then B is diagonalizable.

_____ (ii) If three vectors are linearly dependent, one must be a multiple of another.

_____ (iii) If u is a differentiable function of several variables, then ∇u points in the direction of greatest change.

_____ (iv) Two planes in $\mathbb{R}^3$ intersect in a line.

_____ (v) If $\mathcal{S}$ is a finite set of vectors in $\mathbb{R}^n$, then $(\mathcal{S}^\perp)^\perp = \mathcal{S}$.

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12. (20 Points) State as many conditions as you can think of for a square matrix to be invertible (2 points each, maximum of 10).

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