Solutions to
Midterm Examination II

Math 20
Introduction to Linear Algebra
and Multivariable Calculus

21 April 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
—Handbook for Students

In all situations where row operations are performed on a matrix, label each operation to receive partial credit in case of arithmetic mistakes.

Problem Number	Possible Points	Points Earned
1	15	
2	15	
3	15	
4	15	
5	15	
6	9	
7	16	
Total	100	

1. (15 Points) *Let W be the subspace spanned by*

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix} \right\}$$

Find a basis for W and the dimension of W.

Solution. Notice that $W = \operatorname{Col} A$, where

$$A = \begin{bmatrix} -1 & 2 & 1 & -1 & -2 \\ 2 & -4 & 1 & 5 & 7 \\ 2 & -4 & -3 & 1 & 3 \end{bmatrix}.$$

The reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathbf{r_1}$ and $\mathbf{r_3}$ form a basis $\operatorname{Col} R$, we must have that $\mathbf{a_1}$ and $\mathbf{a_3}$ form a basis for $\operatorname{Col} A$. Thus $\dim W = 2$. $\square$

2. (15 Points) *Find all numbers c such that*

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & c \\ 0 & c & -15 \end{bmatrix}$$

is not invertible.

Solution. By expanding along the bottom row, we see the determinant of this matrix is

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & c \\ 0 & c & -15 \end{vmatrix} = -c \begin{vmatrix} 1 & -1 \\ 2 & c \end{vmatrix} - 15 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= -c(c+2) - 15(-1)$$
$$= -c^2 - 2c + 15$$
$$= -(c-3)(c+5).$$

Thus 3 and -5 are the values of c which make this matrix noninvertible.   □

3. (15 Points) *Let*

$$A = \begin{bmatrix} \frac{11}{10} & \frac{2}{5} \\ \frac{3}{5} & \frac{9}{10} \end{bmatrix}.$$

Consider the discrete dynamical system defined by the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$. *Is* $\mathbf{0}$ *an attractor? repeller? saddle point? Why?*

Solution. We need to find the eigenvalues of A. They are the roots of the polynomial

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} \frac{11}{10} - \lambda & \frac{2}{5} \\ \frac{3}{5} & \frac{9}{10} - \lambda \end{vmatrix} \\ &= \left(\frac{11}{10} - \lambda\right)\left(\frac{9}{10} - \lambda\right) - \frac{6}{25} \\ &= \lambda^2 - 2\lambda + \frac{3}{4}. \end{aligned}$$

The roots of this polynomial are seen to be $\frac{3}{2}$ and $\frac{1}{2}$. Since one of these is bigger than one and one is smaller than one, the origin is a saddle point. $\square$

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4. (15 Points) *Let A be the matrix* $\begin{bmatrix} -2 & -5 & 2 \\ 4 & 7 & -2 \\ -3 & -3 & 5 \end{bmatrix}$. *5 is an eigenvalue of A.*
Give the eigenvector of unit length corresponding to this eigenvalue.

Solution. We must find the parametric form of the general solution to the linear system of equations $(A - 5I)\mathbf{x} = \mathbf{0}$. This can be done by row-reducing $A - 5I$.

$$A - 5I = \begin{bmatrix} -7 & -5 & 2 \\ 4 & 2 & -2 \\ -3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

In the equation $(A - 5I)\mathbf{x} = \mathbf{0}$, x_3 is a free variable, $x_2 = -x_3$, and $x_1 = x_3$.

Hence $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 5. If we scale it

by the reciprocal of its length, we will obtain a unit eigenvector: $\dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. $\square$

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5. (15 Points) *Let*

$$\mathbf{w_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} ; \quad \mathbf{w_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} ; \quad \mathbf{w_3} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} ; \quad \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} .$$

Let $W = \mathrm{Span}\,\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$. *Compute* $\hat{\mathbf{v}} = \mathrm{proj}_W \mathbf{v}$ *and find the distance from* $\mathbf{v}$ *to* W.

Solution. We have

$$\hat{\mathbf{v}} = \frac{\mathbf{v} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_2}} \mathbf{w_1} + \frac{\mathbf{v} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \mathbf{w_2} + \frac{\mathbf{v} \cdot \mathbf{w_3}}{\mathbf{w_3} \cdot \mathbf{w_3}} \mathbf{w_3}$$

$$= \frac{6}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \frac{3}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} .$$

As for the distance between $\mathbf{v}$ and W, it is the length of

$$\mathbf{v} - \hat{\mathbf{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} ,$$

which is $2\sqrt{3}$. ☐

6. (9 Points) *Label the following statements as true or false. Justify your answers. (If true, cite appropriate facts or theorems. If false, explain why the opposite must be true or give a counterexample that shows why the statement is not necessarily true).*

_____ *(i) If A is a matrix whose columns are orthonormal, the linear mapping* $\mathbf{x} \mapsto A\mathbf{x}$ *preserves length.*

Solution. This statement is **true**. Since for all $\mathbf{x}$, $\mathbf{y}$,

$$A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot A^T A\mathbf{y} = \mathbf{x} \cdot \mathbf{y},$$

A preserves the dot product, hence the length of vectors.   □

_____ *(ii) If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.*

Solution. This statement is **false**. Changing the 5 to a -5 would make it true. I didn't expect a counterexample for this problem, although the instructions say to give one. Here is one, however: the matrix $\begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$ has characteristic polynomial $-(\lambda + 5)$, but not 5 as an eigenvalue.

□

_____ *(iii) If A is a square matrix with* $\det A = 0$*, then [at least] one column of A is a multiple of another column of A.*

Solution. This statement is **false**. Most people understood that $\det A = 0$ if and only if the columns of A are linearly dependent. However, there is more to linear dependence than a vector being a multiple of another vector, as most of you know. I suspect that the insertion of "at least" in the problem may have confused more people than it helped, and I'm sorry for that.

Always try to provide a counterexample when you assert something is false. A counterexample is something which satisfies the hypotheses of a proposition but not the conclusions. Consider the matrix

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix}.$$

Its determinant is zero, but no column is a multiple of another. There is a linear dependence relation among the columns, but it is more complicated. (the third is the sum of the first two).   □

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7. (16 Points) *Let A be an n × n matrix which is diagonalizable and whose eigenvalues are all nonnegative (i.e., ≥ 0.) Show there exists an n × n matrix B such that $B^2 = A$.*

Solution. If A is diagonalizable, we may factor it as $A = PDP^{-1}$ where D is diagonal and consists of the eigenvalues of A. Some students went from this equation to the assertion that $A^{1/2} = PD^{1/2}P^{-1}$, but that is putting the cart before the horse, since as yet $A^{1/2}$ is undefined (this is the point of the problem). Since the diagonal entries of D are nonnegative, there is no problem in finding a "square root" matrix F such that $F^2 = D$ (just take the square root of the diagonal entries).

Let $B = PFP^{-1}$. Then

$$B^2 = (PFP^{-1})(PFP^{-1}) = PF^2P^{-1} = PDP^{-1} = A.$$

$\square$