Solutions to Final Examination

Math 20
Introduction to Multivariable Calculus and Linear Algebra

January 23, 2006

Rules:

- This is a three-hour exam.

- Calculators are not allowed.

- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.

- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*

- Please do not put your name on any page besides the first page. If you like, you may put your ID number on the top of each page you write on.

Hints:

- Read the entire exam to scan for obvious typos or questions you might have.

- Budget your time so that you don't run out.

- Problems may stretch across several pages.

- Relax and do well!

   Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
                                        —Handbook for Students

Summary Data

Problem	1	2	3	4	5	6	7	8	9	Total
Maximum Possible	15	15	16	8	10	9	10	9	8	100
Mean	14.90	14.57	11.05	4.29	8.29	7.05	9.38	7.52	7.29	84.33
Median	15	15	14	4	10	9	10	9	8	83
Mode	15	15	15	8	10	9	10	9	8	83
Standard Deviation	0.29	0.90	4.97	2.90	2.58	3.39	1.43	2.06	1.64	12.49
Correlation with total	-0.0563	0.6751	0.8548	0.3789	0.4882	0.7300	0.6620	0.4463	0.5798	1.0000

Part I. Techniques

1. (15 Points) *For $n \times n$ matrices A and B, define*

$$[A, B] = AB - BA.$$

Notice $AB = BA$ if and only if $[A, B] = 0$.
For

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Find

(i) $[H, X]$

Solution. We have

$$HX = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$XH = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\implies [H, X] = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2X.$$

▲

(ii) $[H, Y]$

Solution. Similar to the above we have $[H, Y] = -2Y$.   ▲

(iii) $[X, Y]$

Solution. $[X, Y] = H$.   ▲

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2. (15 Points) *Let*

$$f\begin{pmatrix} x \\ y \end{pmatrix} = 4xy - 2x^4 - y^2$$

(a) Find $\dfrac{\partial f}{\partial x}\begin{pmatrix} x \\ y \end{pmatrix}$ *and* $\dfrac{\partial f}{\partial y}\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution. We have

$$\frac{\partial f}{\partial x}\begin{pmatrix} x \\ y \end{pmatrix} = 4y - 8x^3 = 4(y - 2x^3)$$

$$\frac{\partial f}{\partial y}\begin{pmatrix} x \\ y \end{pmatrix} = 4x - 2y = 2(2x - y).$$

▲

(b) Find the critical points of f.

Solution. We need both partial derivatives to be zero. From the second equation above, we see that we must have $y = 2x$. Hence $2x = 2x^3$ and the possibilities are $x = 0$ or $x = \pm 1$. Hence the critical points are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$,   ▲

(c) For each critical point, decide if it's a local maximum or a local minimum.

Solution. We have

$$\frac{\partial^2 f}{\partial x^2} = -24x^2 \qquad\qquad \frac{\partial^2 f}{\partial x\, \partial y} = 4$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = 4 \qquad\qquad \frac{\partial^2 f}{\partial y^2} = -2$$
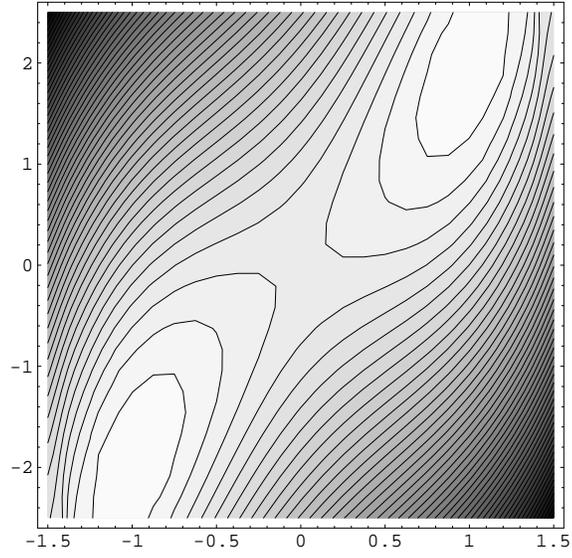
The determinant of the hessian matrix is $48x^2 - 16$, which is positive if $x = \pm 1$. Also at these points the top-left entry is negative. Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ are local maxima. The point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a saddle point.

Here is a contour plot of f:

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3. (16 Points) *Let*

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(a) Is 3 an eigenvalue for A?

Solution. You can just do part (b), but this question can be answered without going to the trouble of computing all the eigenvalues. We simply try to solve the matrix equation $(A - 3I)\mathbf{x} = \mathbf{0}$.

$$A - 3I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So yes, 3 is an eigenvalue for A.   ▲

(b) Find an orthogonal matrix Q and a diagonal matrix Λ such that

$$A = Q\Lambda Q^{\mathsf{T}}$$

Solution. Now we do have to find all the eigenvalues. We have

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2 - \lambda & -1 & -1 \\ -1 & 2 - \lambda & -1 \\ -1 & -1 & 2 - \lambda \end{vmatrix} \\
&= (2 - \lambda)^3 - 2 - (2 - \lambda) - (2 - \lambda) - (2 - \lambda) \\
&= (2 - \lambda)^3 - 8 + 3\lambda \\
&= 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 8 + 3\lambda \\
&= -\lambda(\lambda - 3)^2.
\end{aligned}$$

So the eigenvalues are 0 and 3 (twice).

Now we need the eigenvectors. For the eigenvalue 3, we have already row-reduced $A - 3I$. Apparently x_3 and x_2 are free and $x_1 = -x_2 - x_3$. Therefore the general solution to $(A - 3I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2.$$

Let $\mathbf{v}_1$ be the first vector above and $\mathbf{v}_2$ the second. Since we are eventually in search of an orthogonal matrix, we need to orthogonalize this basis of $\mathbf{E}(2)$. We use the Gram-Schmidt process: Let $\mathbf{u}_1 = \mathbf{v}_1$, and

$$\mathbf{u}_2 = \mathbf{v_2} - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_2}\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

(Orthogonalizing the other way will give you a different basis, but that's OK). Now for the eigenvector corresponding to the eigenvalue 0. We have

$$A - 0I = A \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

so a solution has $x_1 = x_3$ and $x_2 = x_3$. A suitable choice might be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

To form Λ, simply put the eigenvalues on the diagonal, counting with multiplicity. For Q, put the eigenvectors in columns corresponding to their eigenvalues. We have already orthogonalized; we just scale each column to make sure they have norm one. Thus

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \qquad\qquad \Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

▲

4. (8 Points) *Let $W \subset \mathbb{R}^5$ be the subset*

$$W = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ 8 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

Find a basis for W and determine its dimension.

Solution. The usual way to do this is to let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and find its reduced echelon form. We have

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 11 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first two columns are pivot columns, the first two elements of the original set of vectors forms a basis. Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a basis for W. Evidently $\dim W = 2$.

Unfortunately, many tried to find the general solution to $A\mathbf{x} = \mathbf{0}$. This process will give you a basis for $\mathbf{N}(A) = W^\perp$. This is not what was asked, and since the technique used for that problem is different than what we were testing for, we could not give credit for doing the wrong problem. Luckily, however, $\dim W^\perp = \dim W$, so the dimension was accidentally correct. ▲

Part II. Applications

5. (10 Points) *Consider a simple economy consisting of two sectors: goods and services. To produce its output, the goods sector consumes $\frac{3}{4}$ of the goods and $\frac{1}{5}$ of the services. The services sector consumes $\frac{1}{4}$ of the goods and $\frac{4}{5}$ of the services.*

(a) (4 points) Find the exchange matrix A of this economy.

Solution. The products (outputs) form the rows of this matrix and the inputs are the columns. Thus

$$A = \begin{bmatrix} \frac{3}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{4}{5} \end{bmatrix}.$$

▲

(b) (6 points) Find an equilibrium production vector $\mathbf{p}$ that satisfies $A\mathbf{p} = \mathbf{p}$ and $\mathbf{p} \geq \mathbf{0}$ with at least one component positive.

Solution. $A\mathbf{p} = \mathbf{p}$ if and only if $(A - I)\mathbf{p} = \mathbf{0}$, so we row-reduce $A - I$:

$$A - I = \begin{bmatrix} -\frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & -\frac{1}{5} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -\frac{4}{5} \\ 0 & 0 \end{bmatrix}.$$

So a possible value for $\mathbf{p}$ might be $\begin{bmatrix} \frac{4}{5} \\ 1 \end{bmatrix}$, or $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$, or any vector whose components were positive and were in this ratio.   ▲

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6. (9 Points)

I'm getting ready for my annual Super Bowl party and need to order my provisions. Last year my budget was $120, and I spent $40 on nachos and $80 on chicken wings. Economics tells me that I maximized my personal utility function of nachos and wings, which we'll assume is of the form

$$u(x,y) = x^\alpha y^{1-\alpha}.$$

α is some number between 0 and 1. I'm just a consumer, though; I don't know my personal value of α. Can you tell me?

Solution. Here x and y are the dollars spent on nachos and chicken wings (or the number of one-dollar units of each I purchase). No matter what α is, I'm supposed to act in order to maximize my utility subject to the constraint $g(x,y) = x + y = 120$. This means at $(x,y) = (40,80)$, we have

$$\frac{\partial u}{\partial x} = \lambda \frac{\partial g}{\partial x} \qquad\qquad \frac{\partial u}{\partial y} = \lambda \frac{\partial g}{\partial y}$$
$$\alpha x^{\alpha-1} y^{1-\alpha} = \lambda \qquad\qquad (1-\alpha)x^\alpha y^{-\alpha} = \lambda$$

Dividing these two gives

$$\frac{\alpha x^{\alpha-1} y^{1-\alpha}}{(1-\alpha)x^\alpha y^{-\alpha}} = \frac{\lambda}{\lambda}$$
$$\frac{\alpha y}{(1-\alpha)x} = 1$$
$$\frac{\alpha}{1-\alpha} = \frac{x}{y} = \frac{40}{80} = \frac{1}{2}$$
$$2\alpha = 1 - \alpha$$
$$\implies \alpha = \frac{1}{3}.$$

▲

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7. (10 Points) *In designing a new airline route, a company is considering two types of planes, types A and B. Each type A plane can carry 40 passengers and requires two mechanics for servicing; each type B plane can carry 60 passengers and requires three mechanics for servicing. Suppose that the company must transport at least 300 people daily and that insurance rules for the size of the hanger allow no more than 18 mechanics on the payroll. If each type A plane costs $10,000,000 and each type B plane costs $12,000,000, how many planes of each type should be bought to minimize the cost?*

  (i) *(3 points) State the problem in linear programming formalism. What are the decision variables? What is the objective? What are the constraints?*

    Solution. Let x and y be the number of planes of types A and B respectively. We are trying to minimize the total cost of purchase, which is proportional to $10x + 12y$ (no need to write all those extra zeroes!). The constraints are $40x + 60y \geq 300$ (minimum person capacity) and $2x + 3y \leq 18$ (maximum mechanic availability). As always, x and y must be nonnegative. ▲

  (ii) *(3 points) Draw the feasible set.*

    Solution. The equations of constraint in intercept-intercept form are

$$\frac{x}{15/2} + \frac{y}{5} \geq 1$$
$$\frac{x}{9} + \frac{y}{6} \leq 1.$$

▲
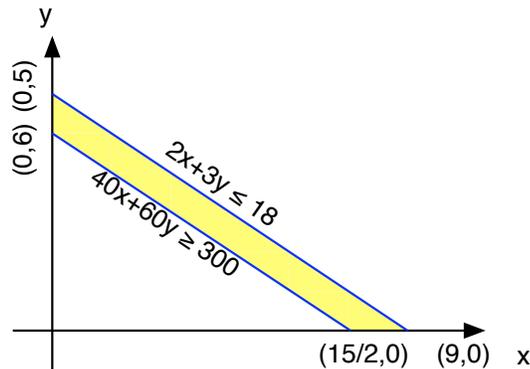
(continued)

(iii) *(4 points) Solve the problem.*

Solution. We need to test the corners.

x	y	cost (millions)
7.5	0	75
9	0	90
0	5	60
0	6	72

Apparently we should buy 5 type B planes and no type A planes. ▲

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8. (9 Points) *Player R has a nickel and dime with him. He chooses one of the coins and player C must guess R's choice. If C guesses correctly, he keeps the coin; if he guesses incorrectly, he must give R an amount equal to the coin shown.*

(a) (3 points) Write down the payoff matrix for this game.

Solution. What determines the payoff is if R and C "choose" the same coin or not. If they do, C wins the value of the coin. If they don't, R wins the value he picked. Thus

$$A = \begin{bmatrix} -5 & 5 \\ 10 & -10 \end{bmatrix}$$

(As usual, these entries represent the payoffs to R).   ▲

(b) (4 points) In repeated rounds of the game how often should each player adopt each strategy?

Solution. We have

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{-20}{-30} = \frac{2}{3}$$

$$p_2 = 1 - p_1 = \frac{1}{3}$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{-15}{-30} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = \frac{1}{2}$$

So R should pick the nickel twice as often as he picks the dime. C should split his choices equally.   ▲

(c) (2 points) Which player does the game favor?

Solution. We have

$$v = \frac{\det A}{a_{11} + a_{22} - a_{21} - a_{12}} = 0,$$

so the game is fair and favors neither player.   ▲

9. (8 Points) *The subscription department of a magazine sends out letters to a large mailing list inviting subscription for the magazine. Some of the people receiving this letter already subscribe to the magazine, while others do not. From this mailing list, 60% of those who already subscribe will subscribe again, while 25% of those who do not now subscribe will subscribe.*

(a) *(3 points) Find the transition matrix A of this Markov process.*

Solution. The columns form the "states" prior to the letter being sent, while the rows are the states afterwards. Let the first row (or column) be the state of subscribing, and the other one not subscribing. We have

$$A = \begin{bmatrix} 3/5 & 1/4 \\ 2/5 & 3/4 \end{bmatrix}.$$

▲

(b) *(5 points) After many letters sent out to the same mailing list, what portion of the members will be subscribed?*

Solution. We need to solve $A\mathbf{q} = \mathbf{q}$, $\mathbf{q} \geq \mathbf{0}$, and $q_1 + q_2 = 1$. For the first condition,

$$A - I = \begin{bmatrix} -2/5 & 1/4 \\ 2/5 & 1/4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -8 & 5 \\ 0 & 0 \end{bmatrix}.$$

Solving $-8q_1 + 5q_2 = 0$ and $q_1 + q_2 = 1$ gives $q_1 = \dfrac{5}{13}$ and $q_2 = \dfrac{8}{13}$.
Therefore $\dfrac{5}{13}$ of the list will eventually be subscribed.   ▲

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