Name: ______________________ ID#: ______________

Solutions to Midterm I

Math 20
Introduction to Linear Algebra and Multivariable Calculus

October 17, 2005

Rules:

- This is a one-hour exam.
- Calculators are not allowed.
- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.
- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*
- Please do not put your name on any page besides the first page. If you like, you may put your ID number on the top of each page you write on.

Hints:

- Read the entire exam to scan for obvious typos or questions you might have.
- Budget your time so that you don't run out.
- Problems may stretch across several pages.
- Relax and do well!

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
—*Handbook for Students*

1. (6 Points) *Let*

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Find

(i) $\mathbf{v} - 2\mathbf{w}$

Solution.

$$\begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$$

▲

(ii) $\mathbf{v} \cdot \mathbf{w}$

Solution. 3 ▲

(iii) $\|\mathbf{v}\|$

Solution. $\sqrt{6}$ ▲

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2. (6 Points) *Let*

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Find AB and BA.

Solution.

$$AB = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

▲

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3. (13 Points)

(i) Let $u\begin{pmatrix}x\\y\\z\end{pmatrix} = x^2y + xy^3 + 3$. *Find* $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, *and* $\frac{\partial u}{\partial z}$.

Solution. We have

$$\frac{\partial u}{\partial x} = 2xy$$
$$\frac{\partial u}{\partial y} = x^2$$
$$\frac{\partial u}{\partial z} = 0$$

▲

(ii) Let $u\begin{pmatrix}x\\y\end{pmatrix} = x^y$. *Find* $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$.

Solution.

$$\frac{\partial u}{\partial x} = yx^{y-1}$$
$$\frac{\partial u}{\partial y} = x^y \ln x$$

▲

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4. (8 Points)

(a) Find the matrix corresponding to the linear transformation which reflects the plane in the x_1 axis. Call this matrix A.

Solution. The matrix corresponding to this linear transformation has columns equal to the image of the standard basis vectors under the transformation. After flipping in the x_1 axis, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is fixed, while $\mathbf{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is taken to its *negative*. Thus

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

▲

(b) Find the matrix corresponding to the linear transformation which reflects the plane in the x_2 axis. Call this matrix B.

Solution. Under this transformation, $\mathbf{e}_1$ is flipped while $\mathbf{e}_2$ is reflected. Thus

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

▲

(c) Find the matrix corresponding to the linear transformation which rotates the plane by $180°$ about the origin. Call this matrix C.

Solution. Here both $\mathbf{e}_1$ and $\mathbf{e}_2$ are flipped. Therefore

$$C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

▲

(d) What is the product AB?

Solution. Multiplying it out, we see that $AB = C$. It's a general fact that the product of two reflections is always a rotation. ▲

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5. (6 Points) *Let V be the following subset of $\mathbb{R}^3$:*

$$V = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; \mathbf{x} = \begin{bmatrix} s+t \\ 2s+t \\ s+2t \end{bmatrix} \text{ where } s, t \in \mathbb{R} \right\}$$

We will show that V is a subspace of $\mathbb{R}^3$ in two ways:

(i) Show that V is the span of two vectors: In other words, there exist vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ such that

$$V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

Solution. The vector $\mathbf{x} \in V$ if and only if

$$\mathbf{x} = s\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Then if $\mathbf{v}_1$ and $\mathbf{v}_2$ are the two vectors above, $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. ▲

(ii) Show that V is the "perp space" to a vector $\mathbf{w}$: In other words, there exists a vector $\mathbf{w} \in \mathbb{R}^3$ such that

$$V = \mathbf{w}^\perp = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; \mathbf{x} \cdot \mathbf{w} = \mathbf{0} \right\}.$$

Solution. The perp space to V is the set of all vectors $\mathbf{w}$ such that $\mathbf{w}\cdot\mathbf{v}_1 = 0$ and $\mathbf{w} \cdot \mathbf{v}_2 = 0$. So $\mathbf{w}$ is a solution to the system of linear equations that has coefficients

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Gaussian elimination reduces this to $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$, which gives the general solution of x_3 free, $x_2 = x_3$, and $x_1 = -3x_3$. So

$$\mathbf{w} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

generates $V^\perp$. By taking the orthogonal complements (perp spaces) to both of these, we have $V = \mathbf{w}^\perp$. ▲

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6. (11 Points)

(i) Show

$$\lim_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \frac{x}{x^2 + y^2}$$

does not exist by finding a line through the origin along which the limit does not exist.

Solution. Take the path through the origin along the x-axis, so $y = 0$. Thus

$$\lim_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \frac{x}{x^2 + y^2} = \lim_{x \to 0} \frac{1}{x},$$

which does not exist. ▲

(ii) Show

$$\lim_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \frac{xy}{x^2 + y^2}$$

does not exist by finding two lines through the origin along which the limits exist but disagree.

Solution. A line through the origin (except for the y-axis) takes the form $y = mx$. Substituting this into the limit gives

$$\lim_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x(mx)}{x^2 + (mx)^2} = \frac{m}{1 + m^2} \lim_{x \to 0} \frac{1}{2} = \frac{m}{1 + m^2}.$$

If $m = 1$, the limit is $\frac{1}{2}$, and if $m = -1$, the limit is $-\frac{1}{2}$. ▲

(iii) Now we will show

$$\lim_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} \frac{xy^2}{x^2 + y^2} = 0$$

(a) Show that

$$|x| \leq \sqrt{x^2 + y^2} \qquad\qquad |y| \leq \sqrt{x^2 + y^2}$$

for all real numbers x and y. (This is easier than it looks; you don't need any fancy theorems. Why not square both sides?)

Solution. Clearly $x^2 \leq x^2 + y^2$ for all x and y, because any number squared is no worse than zero. Taking the square root of this inequality gives us what we want. ▲

(b) Use (a) to show that

$$\left|\frac{xy^2}{x^2+y^2}\right| \leq \sqrt{x^2+y^2}.$$

(You can do this part even if you didn't get part (a).)

Solution. We have

$$\left|\frac{xy^2}{x^2+y^2}\right| \leq \frac{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}\right)^2}{x^2+y^2} = \sqrt{x^2+y^2}.$$

▲

(c) What is the limit of the right-hand side of this inequality as x and y both tend to zero? (Just say the answer, no need to prove it.)

Solution. The limit is zero. Thus $\left|\frac{xy^2}{x^2+y^2}\right|$ is trapped between 0 and $\sqrt{x^2+y^2}$, both of which have a limit of zero as x and y tend to zero. Therefore

$$\lim_{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\longrightarrow\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} \left|\frac{xy^2}{x^2+y^2}\right| = 0 \implies \lim_{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]\longrightarrow\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} \frac{xy^2}{x^2+y^2} = 0$$

▲

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