Name: ______________________________ ID#: ____________________

Solutions to Midterm III

Math 20
Introduction to Linear Algebra and Multivariable Calculus

December 16, 2005

Rules:

- This is a one-hour exam.
- Calculators are not allowed.
- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.
- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*
- Please do not put your name on any page besides the first page. If you like, you may put your ID number on the top of each page you write on.

Hints:

- Read the entire exam to scan for obvious typos or questions you might have.
- Budget your time so that you don't run out.
- Problems may stretch across several pages.
- Relax and do well!

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
—Handbook for Students

1. (5 Points) *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation whose standard matrix is*

$$A = [T]_{\mathcal{E}} = \begin{bmatrix} 2 & 8 \\ 4 & -2 \end{bmatrix}$$

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find $[T]_{\mathcal{B}}$.

Solution. Let P be the matrix whose columns are the elements of $\mathcal{B}$ in the coordinates of $\mathcal{E}$. That is,

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned} [T]_{\mathcal{B}} = P^{-1}AP &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 \\ 4 & -6 \end{bmatrix}. \end{aligned}$$

□

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2. (5 Points) *It is March 1943, and Rear Admiral Kimura Masatomi is trying to move a convoy from Rabaul, on the northeastern tip of the island of New Britain, to Lae, just west of New Britain on the island of New Guinea.*

He can take the northern route or the southern route around New Britain. Both routes would take three days to reach Lae but visibility will be poor along the northern route due to bad weather.

The Allies have learned of this plan (but not Masatomi's choice). Their strategy is to find the convoy by air reconnaissance and bomb it until it reaches Lae. They need to decide whether to search along the northern route or along the southern route. Clearly they will have more time to bomb if they choose correctly, and remember the visibility along the northern route is poor. General George Kenney's staff estimate that depending on the routes taken, the possible number of days of bombing can be given by this table:

		Japanese	
		North	*South*
Allies	*North*	*2*	*2*
	South	*1*	*3*

What should Kenney and Masatomi decide and what will be the result?

Solution. For the Japanese admiral, the northern route dominates the southern route (his losses are no worse). So the game reduces to $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$. Now for the Allied general, the northern route dominates the southern route, too. So both parties should go north of the island.

This is exactly what happened. The result became known as the Battle of the Bismarck Sea, and was a decisive victory for the Allies. The Japanese lost eight destroyers, eight troop transports, about 100 fighter planes, and at least 2,890 soldiers. The game was unfair from the start, but Masatomi played it well. □

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3. (10 Points) *During a penalty kick in soccer, the shooter S kicks the ball towards the goal while the goalie G defends the goal.*

S may aim to the left or to the right of the goal, G may dive to his left or right. (So if S aims left, and G dives right, ball and goalkeeper should be close together.) Suppose G has a 50% chance of stopping a shot on target if he dives the way that S kicks. Further suppose that the kicker is more accurate when shooting to the right—he is then on target 90% of the time, but only 70% of the time when shooting left.

(a) Find the payoff matrix for this game.

Solution. The payoff is the probability of a successful (on target and not defended) goal. So if the shooter aims left, his payoff is 70% or 35% depending on whether the goalie guesses correctly. If the shooter aims right, his payoff is 90% or 45%. Therefore the payoff matrix is

		Goalie	
		Left	Right
Shooter	Left	70	35
	Right	45	90

□

(b) With what probability should the kicker decide to shoot to the left?

Solution. This is a 2×2 game with no dominant strategies, so we use the formula

$$p_1 = \frac{90 - 45}{70 + 90 - 45 - 35} = \frac{45}{80} = 56.25\%.$$

So the shooter should go *left* (kind of surprising since this is his weak side) slightly more often. □

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4. (15 Points) *In the back room of a book store there are a large number of copies of books. There is a novel by Nabokov and one by Updike. There are also calculus books: one by Stewart, one by Thomas, and one by Gottlieb. The novels are each one inch thick, and the calculus books are each two inches thick.*

Let r_n be the number of ways of arranging books in a stack n inches high.

(i) Find r_0, r_1, and r_2.

(ii) Show that $\{r_n\}$ satisfies the difference equation

$$r_n = 2r_{n-1} + 3r_{n-2}$$

for $n \geq 2$.

(iii) Solve for r_n in terms of n.

(iv) How many ways are there to stack the books three feet high?

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5. (15 Points) *A department is trying to schedule teaching fellows for a course. The teaching fellows are polled and have given their preferences of section time. TF A prefers teaching at 9* AM, *10* AM, *11* AM, *12* PM *in that order. TF B prefers teaching at 10, 12, 11, and 9. TF C prefers 10, 9, 12, and 11.*

Find an assignment of teaching fellows to section times that maximizes satisfaction. (Note: you will have to invent a dummy TF who will take the leftover time slot.)

Solution. The Hungarian method is for minimizing costs. If we make the "cost" of an assignment that TF's preference rank, then maximum satisfaction equals minimum total cost. We invent a dummy (so to speak) TF who's completely indifferent. We can make a cost matrix that looks like this:

	A	B	C	D
9 : 00	1	4	2	0
10 : 00	2	1	1	0
11 : 00	3	3	4	0
12 : 00	4	2	3	0

The first step is to subtract the minimum entry from every row, but this is already done since the minimum entry is zero. Then we do the same with the colums, subtracting 1 from each of the first three columns to normalize them (notice that if we had chosen a different value for each of TF D's preferences, this step would zero out that column anyway).

	A	B	C	D
9 : 00	0	3	1	0
10 : 00	1	0	0	0
11 : 00	2	2	2	0
12 : 00	3	(1)	2	0

As you can see, we can cover all the zeros with only 3 lines, so we need to make further adjustments to get an assignment of zeros. The minimum entry is 1; we subtract that from all uncovered entries and add it to all double-covered entries.

	A	B	C	D
9 : 00	(0)	3	1	1
10 : 00	1	0	(0)	1
11 : 00	1	1	1	(0)
12 : 00	2	(0)	1	0

After this adjustment, we have evidently found an assignment of zeros. This corresponds to giving the 9:00 section to TF A, the noon section to TF B, and the 10:00 section to TF C. TF D (nobody) will teach the 11:00 section. □

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