Solutions to Final Examination

Math 20
Introduction to Multivariable Calculus and Linear Algebra

May 19, 2006

Summary Data

Problem	1	2	3	4	5	6	7
Max Possible	9	9	10	10	10	10	10
Max Achieved	9	9	10	10	10	10	10
Mean	8.89	8.75	9.03	9.44	9.69	9.81	9.22
Median	9.0	9.0	9.0	10.0	10.0	10.0	10.0
Mode	9	9	9	10	10	10	10
% full credit	90%	93%	40%	80%	90%	87%	70%
% no credit	0%	0%	0%	0%	0%	0%	0%
Standard Deviation	0.3928	1.0897	0.9570	1.8325	0.9667	0.4606	1.8725
Correlation with Total	0.0773	0.7614	0.6707	0.4646	0.6925	0.5538	0.7940

Problem	8	9	10	11	12	Total	Percent
Max Possible	10	10	12	10	10	120	100.00%
Max Achieved	10	10	12	10	7	117	97.50%
Mean	8.94	8.89	8.83	5.14	3.83	100.47	83.73%
Median	9.0	10.0	10.0	5.0	4.0	102.5	85.42%
Mode	10	10	12	5	6	107	89.17%
% full credit	37%	63%	23%	13%	0%	0%	0%
% no credit	0%	0%	3%	7%	13%	0%	0%
Standard Deviation	0.9985	1.7760	3.7896	2.6890	2.0480	11.7933	9.83%
Correlation with Total	0.3349	0.7677	0.7227	0.5918	0.4886	1.0000	1.0000

1. (9 Points) *Let*

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix}$$

Find

(i) BA

Solution. The answer is

$$BA = \begin{bmatrix} -1 & 3 \\ -1 & 13 \\ -3 & 4 \end{bmatrix}.$$

▲

(ii) AB^T

Solution. The answer is

$$AB^T = \begin{bmatrix} 2 & 8 & 3 \\ 3 & 7 & 7 \end{bmatrix}.$$

▲

(iii) A^2

Solution. The answer is

$$A^2 = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix}.$$

▲

2. (9 Points) *Let*

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Find

(a) $\mathbf{v} - 3\mathbf{w}$

Solution. The answer is $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$. ▲

(b) $\mathbf{v} \cdot \mathbf{w}$

Solution. The answer is $0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 1$. ▲

(c) $\|\mathbf{w}\|$.

Solution. The answer is $\sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$. ▲

3. (10 Points)

I. (6 points) Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

Find the reduced row echelon form of A.

Solution. We have

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \begin{matrix} | \cdot (-1) \\ \\ \leftarrow + \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \begin{matrix} \\ | \cdot \frac{1}{2} \\ \\ \\ \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \begin{matrix} \\ | \cdot (-1) \\ \\ \leftarrow + \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ | \cdot (-1) \\ \leftarrow + \end{matrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ | \cdot (-1) \\ \leftarrow + \end{matrix}$$

▲

(continued)

II. *(4 points) Suppose the system of linear equations $A\mathbf{x} = \mathbf{b}$ has an augmented matrix whose reduced row echelon form is*

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the parametric form to the general solution.

Solution. We have five variables:

$$\begin{aligned} &x_5 \quad \text{is free} \\ &x_4 = 3 - 2x_5 \\ &x_3 \quad \text{is free} \\ &x_2 = 2 - x_5 + x_3 \\ &x_1 = 4 + x_4 - 2x_3 \end{aligned}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 - 2x_3 + x_5 \\ 2 - x_5 + x_3 \\ x_3 \\ 3 - 2x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

▲

4. (10 Points) *As in Problem 3, let*

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 \end{bmatrix}.$$

Find the determinant of A.

Solution. We expand by cofactors down the first column, getting

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 \end{vmatrix} &= \begin{vmatrix} 2 & 4 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \\ &= 8 - 4 - 4 = 0. \end{aligned}$$

▲

An even easier solution is this: Two of the many conditions that a matrix be invertible are that its reduced row echelon form is the identity matrix and that its determinant is nonzero. By Problem 3, the first condition is false, so the determinant of *A* must be zero.

5. (10 Points) *Let*

$$A = \begin{bmatrix} 13 & -8 \\ 8 & -7 \end{bmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}.$$

Solution. First, we need to find the eigenvalues of A. They are the roots of the characteristic polynomial:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 13 & 8 \\ -8 & \lambda + 7 \end{vmatrix} = (\lambda - 13)(\lambda + 7) + 64$$
$$= \lambda^2 - 6\lambda - 27 = (\lambda - 9)(\lambda + 3)$$

So the eigenvalues are 9 and -3, telling us that $D = \begin{bmatrix} 9 & 0 \\ 0 & -3 \end{bmatrix}$.

For each λ, an eigenvector spans the null space of $A - \lambda I$. For the eigenvalue $\lambda = 9$, we have

$$A - 9I = \begin{bmatrix} 4 & -8 \\ 8 & -16 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

So $x_1 = 2x_2$ and an eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. For the eigenvalue $\lambda = -3$, we have

$$A + 3I = \begin{bmatrix} 16 & -8 \\ 8 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

So $x_1 = \frac{1}{2}x_2$ and an eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The matrix P needs to have eigenvectors for columns. Thus

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

▲

6. (10 Points) *Let*

$$u(x,y) = xy^3 - x^3y$$

Find

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Solution. We have

$$\frac{\partial u}{\partial x} = y^3 - 3x^2y \qquad \frac{\partial u}{\partial y} = 3xy^2 - x^3$$

$$\frac{\partial^2 u}{\partial x^2} = -6xy \qquad \frac{\partial^2 u}{\partial y^2} = 6xy$$

These sum to zero. ▲

7. (10 Points) *A gourmand is deciding on what to have for dessert. He likes pie by itself and pie with ice cream, but not ice cream by itself. His utility function could be*

$$u(x, y) = xy + 2x,$$

where x is number of slices of pie and y is the number of scoops of ice cream ordered.

Slices of pie cost \$2 and scoops of ice cream cost \$1, and the gourmand has \$19 to spend on dessert. What should he order to maximize his utility (and his waistline, probably!).

Solution. The objective is to maximize u subject to the constraint that

$$g(x, y) = 2x + y = 19.$$

By the method of Lagrange multipliers

$$\frac{\partial u}{\partial x} = \lambda \frac{\partial g}{\partial x} \implies y + 2 = 2\lambda$$
$$\frac{\partial u}{\partial y} = \lambda \frac{\partial g}{\partial y} \implies x = \lambda$$

Combining these gives $y + 2 = 2x$, or $-2x + y = -2$. Combining these with $2x + y = 19$ gives $x = \frac{21}{4} = 5.25$, $y = \dfrac{17}{2} = 8.5$. ▲

8. (10 Points) *A pharmaceutical company is creating a tablet for a new drug. Each tablet is to contain a binder, a disintegrant and a filler in addition to the active drug ingredient, which is to be 14% of the weight of each tablet. Chemical and physical considerations mean that the weight of the disintegrant should not exceed 25% of the combined weights of the binder and the active ingredient, and that there should be at most 10 times as much filler as binder. The disintegrant costs $15, the binder $50 and filler $2 per kilogram.*

(i) *Formulate a linear programming problem which shows the mixture of ingredients in a 100kg lot that minimizes costs but satisfies all chemical and physical considerations. (It won't be in standard form.)*

Solution. We have three decision variables: y_1 for the amount of binder, y_2 the amount of disintegrant, and y_3 the amount of filler. The objective function is to minimize

$$w = 50y_1 + 15y_2 + 2y_3.$$

One constraint is that, since the active ingredient is to make up 14 kg of the lot, the remaining ingredients must weigh 86 kg:

$$y_1 + y_2 + y_3 = 86.$$

The other two constraints are

$$\begin{aligned} y_2 &\leq \tfrac{1}{4}(y_1 + 14) &&\implies -\tfrac{1}{4}y_1 + y_2 \leq \tfrac{14}{4} \\ y_3 &\leq 10y_1 &&\implies -10y_1 + y_3 \leq 0 \end{aligned}$$

▲

(ii) *Formulate the dual problem. You need not solve either.*

Solution. The naïve "solution" is the following: We have three decision variables x_1, x_2, and x_3, and the new objective is to maximize

$$z = 86x_1 + \tfrac{14}{4}x_2$$

subject to constraints

$$\begin{aligned} x_1 - \tfrac{1}{4}x_2 - 10x_3 &= 50 \\ x_1 + x_2 &\geq 15 \\ x_1 + x_3 &\geq 2 \end{aligned}$$

I gave full credit for this work. However, it's completely wrong, as I realized while grading it. This LP problem has no solution, though; w can be made infinitely large without leaving the feasibility set. That reminded me that the "dualization" process only works for problems in standard form (or those which are already duals of problems in standard form). This means we have to standardize the primal problem.

The equality constraint can be eliminated by solving for y_3: $y_3 = 86 - y_1 - y_2$. Then w becomes

$$w = 50y_1 + 15y_2 + 2(86 - y_1 - y_2) = 48y_1 + 13y_2 + 172.$$

The point (y_1, y_2) which minimizes this function will also minimize $w' = 48y_1 + 13y_2$, so this could also be our objective. Let's put this problem in the form of the dual problem to a standard form problem. We just have to change the constraints to $\geq$ constraints.

The second constraint is already only in terms of y_1 and y_2, and is a $\leq$ constraint. To make it a $\geq$ constraint, multiply by -1:

$$\frac{1}{4}y_1 - y_2 \geq -\frac{14}{4}.$$

$$-10y_1 + (86 - y_1 - y_2) \leq 0 \implies -11y_1 - y_2 \leq -86 \implies 11y_1 + y_2 \geq 86.$$

The dual problem is then to maximize

$$z = -\frac{14}{4}x_1 + 86x_2$$

subject to constraints

$$\begin{aligned} \frac{1}{4}x_1 + 11x_2 &\leq 48 \\ -x_1 + x_2 &\leq 13 \end{aligned}$$

You can check that both of these problems have the same optimal values. ▲

9. (10 Points) *Consider a planet consisting of two countries, Redsoxnation and Yankeeland. The total income of each is derived from consumption of local goods and importation of foreign goods. Luckily, they share a common currency.*

Each year in Redsoxnation, 70 cents from every dollar is spent on Redsoxnation goods and the rest on foreign goods. In Yankeeland, it's 90 cents from every dollar that is spent locally. The next year each country spends their income according to the same ratio.

What is the eventual distribution of money between the two countries?

Solution. We can use a Markov chain with transition matrix:

$$A = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix}$$

It's very important to have the columns of the transition matrix add up to one, and to keep the states the same in rows and columns. Common mistakes were to use the transpose of A instead of A, or to divide the final states into local and foreign goods. Remember that local goods in Redsoxnation mean dollars spent in Redsoxnation, and the rest are imports from Yankeeland, etc.

The eventual distribution of money is a steady-state vector $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$. It solves the equations $A\mathbf{p} = \mathbf{p}$ and $p_1 + p_2 = 1$.

To find $\mathbf{p}$, we reduce

$$A - I = \begin{bmatrix} -0.3 & 0.1 \\ 0.3 & -0.1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus $-3p_1 + p_2 = 1$. Combining in the second equation $p_1 + p_2 = 1$, we get $p_1 = \frac{1}{4}$ and $p_2 = \frac{3}{4}$. Thus $\frac{1}{4}$ of the money ends up in Redsoxnation. ▲

10. (12 Points) *Suppose that a force y is applied to one end of a spring that has its other end fixed, thus stretching it to a certain length x. In physics,* Hooke's Law *states that (within certain limits), there is a linear relation between x and y. That is, there are constants a and b such that $y = a + bx$. The coefficient b is called the* spring constant.

An experiment is done which measures the force y in a spring at certain levels of displacement x. The data are:

Length	*Force*
1	*14*
3	*17*
5	*19*
7	*20*

What is the spring constant?

Solution. x We are looking for a solution to the system of linear equations

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 14 \\ 17 \\ 19 \\ 20 \end{bmatrix}.$$

Of course, this system is inconsistent. So we instead try to solve the normal equations related to this system, by multiplying by the transpose of the matrix on the left. We get

$$\begin{bmatrix} 4 & 16 \\ 16 & 84 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 70 \\ 300 \end{bmatrix}.$$

Solving this reveals $b = 1$ and $m = \frac{27}{2}$. So the spring constant is 13.5. ▲

11. (10 Points) *Let A be an arbitrary 3×2 matrix, and $\mathbf{b}$ an arbitrary vector in $\mathbb{R}^3$.*

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} g \\ h \\ k \end{bmatrix}$$

Use calculus to show that if (x, y) minimizes the function

$$f(x,y) = \left\| A \begin{bmatrix} x \\ y \end{bmatrix} - \mathbf{b} \right\|^2,$$

then the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfies

$$A^T A \begin{bmatrix} x \\ y \end{bmatrix} = A^T \mathbf{b}.$$

Solution. Multiplying it out we have

$$f(x,y) = (ax + by - g)^2 + (cx + dy - h)^2 + (ex + fy - k)^2.$$

The derivatives of f are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2(ax + by - g)a + 2(cx + dy - h)c + 2(ex + fy - k)e \\ \frac{\partial f}{\partial x} &= 2(ax + by - g)b + 2(cx + dy - h)d + 2(ex + fy - k)f. \end{aligned}$$

The equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ give

$$\begin{aligned} (a^2 + c^2 + e^2)x + (ab + cd + ef)y &= ag + eh + ek \\ (ab + cd + ef)x + (b^2 + d^2 + f^2)y &= bg + dk + fk \end{aligned}$$

We don't have to solve these equations. We just have to show they're equivalent to $A^T A \begin{bmatrix} x \\ y \end{bmatrix} = A^T \mathbf{b}$. In matrix form these equations are

$$\begin{bmatrix} a^2 + c^2 + e^2 & ab + cd + ef \\ ab + cd + ef & b^2 + d^2 + f^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ag + eh + ek \\ bg + dk + fk \end{bmatrix}$$

or

$$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \begin{bmatrix} g \\ h \\ k \end{bmatrix}.$$

In other words, $A^T A \begin{bmatrix} x \\ y \end{bmatrix} = A^T \mathbf{b}$.

Inspection of the function shows that it has a unique global minimum, and so the only critical point is the minimum, so no further testing is needed. However, it's just a bit of algebra to show that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 2(a^2 + c^2 + e^2) \\ \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \, \partial y} \right) &= 4 \left\{ (ad - bc)^2 + (af - be)^2 + (cf - de)^2 \right\}, \end{aligned}$$

so in most cases the second derivative test tells you this is a minimum as well.

There is a way to take derivatives and leave the equations as vector equations, but because multiplication of matrices is not always defined nor is it usually commutative, some care has to be taken. Transposes and odd orders of multiplication appear where you don't see them in one-variable calculus. It's a notational aid that's almost as hard to remember as it is to do the algebra on the level of entries. But, if

$$f(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|^2 = (A\mathbf{x} - \mathbf{b})^T(A\mathbf{x} - \mathbf{b})$$

Then

$$\frac{df}{d\mathbf{x}} = 2(A\mathbf{x} - \mathbf{b})^T A = 2(\mathbf{x}^T A^T A - \mathbf{b}^T A)$$

So $\frac{df}{d\mathbf{x}} = \mathbf{0}$ whenever

$$\mathbf{x}^T A^T A = \mathbf{b}^T A \iff A^T A\mathbf{x} = A^T\mathbf{b}.$$

▲

12. (10 Points) *Let A be an arbitrary 3×3 symmetric matrix.*

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Let

$$f(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Show that if (x,y,z) is a critical point of f on the unit sphere $x^2+y^2+z^2=1$, then $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an eigenvector for A associated to the eigenvalue $f(x,y,z)$.

Solution. The function simplifies to

$$f(x,y,z) = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$$

and we need to find the critical points of this subject to the equation of constraint $x^2+y^2+z^2=1$. The lagrange multiplier equations give us

$$\begin{aligned} 2ax + 2by + 2cz &= \lambda(2x) \\ 2bx + 2dy + 2ez &= \lambda(2y) \\ 2cx + 2ey + 2fz &= \lambda(2z) \end{aligned}$$

Dividing out the 2 and rewriting as a matrix equation gives

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

which says that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an eigenvector for A associated to the eigenvalue λ. Once again, finding the value of x, y, and z is unnecessary.

Moreover, if x, y, and z are as above,

$$f(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda(x^2+y^2+z^2) = \lambda.$$

So each critical point of f is an eigenvector, and the associated critical value its eigenvalue. ▲