Name: ______________________________ ID#: ____________________

Solutions to Midterm I

Math 20
Introduction to Multivariable Calculus and Linear Algebra

March 10, 2006

Rules:

- This is a one-hour exam.
- Calculators are not allowed.
- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.
- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*
- Please do not put your name on any page besides the first page. If you like, you may put your ID number on the top of each page you write on.

Hints:

- Read the entire exam to scan for obvious typos or questions you might have.
- Budget your time so that you don't run out.
- Problems may stretch across several pages.
- Relax and do well!

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.
—*Handbook for Students*

Summary Data

Problem	1	2	3	4	5	Total	Percent
Maximum Possible	9	15	15	11	10	60	100%
Maximum Achieved	9	15	15	11	10	60	100%
Mean	8.11	14.33	14.06	9.56	7.53	53.58	89.31%
Median	9	15	15	10	8	55	92%
Mode	9	15	15	10	9	58	97%
% full credit	70%	80%	60%	23%	3%	3%	3%
% no credit	0%	0%	0%	0%	0%	0%	0%
Standard Deviation	1.7123	2.0000	1.4897	1.9213	2.2046	6.8369	11.39%
Correlation with Total	0.7561	0.8390	0.7032	0.6097	0.7462	1.0000	100%

1. (9 Points) *Let*

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Find

(i) $2\mathbf{v} - \mathbf{w}$

Solution. We have

$$2\mathbf{v} - \mathbf{w} = \begin{bmatrix} 2 \cdot 0 - 2 \\ 2 \cdot 1 - 1 \\ 2 \cdot 3 - 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$$

▲

(ii) $\mathbf{v} \cdot \mathbf{w}$

Solution. We have

$$\mathbf{v} \cdot \mathbf{w} = 0 \cdot 2 + 1 \cdot (-1) + 3 \cdot 0 = -1.$$

▲

(iii) $\|\mathbf{v}\|$.

Solution. We have

$$\|\mathbf{v}\| = \sqrt{0^2 + 1^2 + 3^2} = \sqrt{10}$$

▲

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2. (15 Points) *For $n \times n$ matrices A and B, define*

$$[A, B] = AB - BA.$$

Notice $AB = BA$ if and only if $[A, B] = 0$.
For

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Find

(i) $[H, X]$

Solution. We have

$$\begin{aligned} HX &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ XH &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \\ \implies [H, X] &= \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2X. \end{aligned}$$

▲

(ii) $[H, Y]$

Solution. Similar to the above we have $[H, Y] = -2Y$. ▲

(iii) $[X, Y]$

Solution. $[X, Y] = H$. ▲

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3. (15 Points) *Let*

$$f(x,y) = 4xy - 2x^4 - y^2$$

(a) Find $\dfrac{\partial f}{\partial x}(x,y)$ *and* $\dfrac{\partial f}{\partial y}(x,y)$.

Solution. We have

$$\frac{\partial f}{\partial x}(x,y) = 4y - 8x^3 = 4(y - 2x^3)$$
$$\frac{\partial f}{\partial y}(x,y) = 4x - 2y = 2(2x - y).$$

▲

(b) Find the critical points of f.

Solution. We need both partial derivatives to be zero. From the second equation above, we see that we must have $y = 2x$. Hence $2x = 2x^3$ and the possibilities are $x = 0$ or $x = \pm 1$. Hence the critical points are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$, ▲

(c) For each critical point, decide if it's a local maximum or a local minimum (or neither).

Solution. We have

$$\frac{\partial^2 f}{\partial x^2} = -24x^2 \qquad \frac{\partial^2 f}{\partial x\,\partial y} = 4$$
$$\frac{\partial^2 f}{\partial y\,\partial x} = 4 \qquad \frac{\partial^2 f}{\partial y^2} = -2$$

The determinant of the hessian matrix is $48x^2 - 16$, which is positive if $x = \pm 1$. Also at these points the top-left entry is negative. Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ are local maxima. The point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a saddle point.

Here is a contour plot of f:

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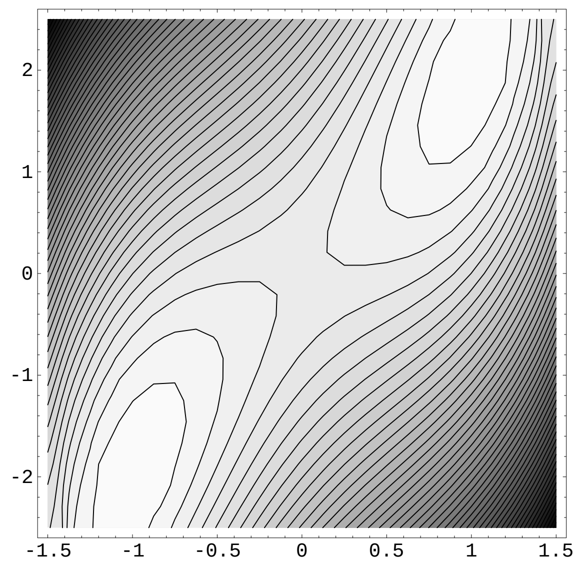

▲

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4. (11 Points) *Consider a production function of the form*

$$P(K,L) = C\left(sK^{-\alpha} + (1-s)L^{-\alpha}\right)^{-1/\alpha},$$

where K is capital invested and L is labor employed per unit time. Here α is a constant greater than one and s is a constant between zero and one. C is another constant to rectify units.

Find and simplify the quantity

$$\sigma = \frac{P_K P_L}{P\,P_{KL}}.$$

Hint. Just take the derivatives. The expression σ is known as the *elasticity of substitution* and a function of this form is known as a *constant elasticity of substitution function.*

Solution. Let $U = sK^{-\alpha} + (1-s)L^{-\alpha}$, so that $P = CU^{-1/\alpha}$. This will make things a little more cleaner. We have

$$\begin{aligned} P_K = \frac{\partial P}{\partial K} &= C\left(-\tfrac{1}{\alpha}\right)U^{-1/\alpha-1}s(-\alpha)K^{-\alpha-1} \\ &= CsU^{-1/\alpha-1}K^{-\alpha-1} \\ P_L = \frac{\partial P}{\partial L} &= C\left(-\tfrac{1}{\alpha}\right)U^{-1/\alpha-1}s(-\alpha)K^{-\alpha-1} \\ &= C(1-s)U^{-1/\alpha-1}L^{-\alpha-1} \\ P_{KL} = \frac{\partial^2 P}{\partial L\,\partial K} &= Cs\left(-\tfrac{1}{\alpha}-1\right)U^{-1/\alpha-2}K^{-\alpha-1}L^{-\alpha-1} \\ &= Cs(1-s)(1+\alpha)U^{-1/\alpha-2}K^{-\alpha-1}L^{-\alpha-1}. \end{aligned}$$

Putting this together, we have

$$\sigma = \frac{P_K P_L}{P\,P_{KL}} = \frac{\left(CsU^{-1/\alpha-1}K^{-\alpha-1}\right)\left(C(1-s)U^{-1/\alpha-1}L^{-\alpha-1}\right)}{\left(CU^{-1/\alpha}\right)\left(Cs(1-s)(1+\alpha)U^{-1/\alpha-2}K^{-\alpha-1}L^{-\alpha-1}\right)}$$

We get lots of cancelations—the factors of C, s, $1-s$, $K^{-\alpha-1}$, and $L^{-\alpha-1}$. The power of U in the numerator is $-\frac{1}{\alpha} - 1 - \frac{1}{\alpha} - 1 = -\frac{2}{\alpha} - 2$, and the power of U in the denominator is $-\frac{1}{\alpha} - \frac{1}{\alpha} - 2$, which is the same thing. Hence *everthing* cancels except

$$\sigma = \frac{1}{1+\alpha},$$

a constant. ▲

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5. (10 Points) *A student derives utility (happiness) from consuming burritos and chips according to the function*

$$u(x, y) = 10x^{1/2}y^{2/5},$$

where x is the number of burritos consumed and y is the number of baskets of chips consumed every week. Burritos cost \$2 apiece and baskets of chips cost \$3 apiece.

(i) *(8 points) If the student has a fixed budget of \$60 with which to buy his Mexican food, what quantities of burritos and chips should he buy to maximize his utility?*

Solution. We need to maximize u subject to the constraint that

$$g(x, y) = 2x + 3y = 60.$$

Using a Lagrange multiplier λ, we have

$$\frac{\partial u}{\partial x} = \lambda\frac{\partial g}{\partial x} \implies 5x^{-1/2}y^{2/5} = 2\lambda$$
$$\frac{\partial u}{\partial y} = \lambda\frac{\partial g}{\partial y} \implies 4x^{1/2}y^{-3/5} = 3\lambda$$

Dividing the two equations, we get

$$\frac{5y}{4x} = \frac{2}{3} \implies 8x = 15y.$$

Solving the two equations $2x + 3y = 60$, $8x - 15y = 0$ gives $x = \frac{50}{3}$ and $y = \frac{80}{9}$.

We conveniently ignore the fact that these numbers are not integers. In a suitably large scaling of the problem we should apportion our resources by this ratio. ▲

(ii) (2 points) Suppose the restaurant offers the student an incentive: one free burrito or one free basket of chips. Which should he take?

Solution. The question can be answered by finding the marginal utility of burritos and chips. That is, we want to compute $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ at this point of utility maximization. That would involve plugging in $x = \frac{50}{3}$ and $y = \frac{80}{9}$, except remember that according to the Lagrange multiplier equations, one of these is 2λ and one of them is 3λ. It's easiest to see which between these is bigger!

The economists answer it this way (I think): The power on x is higher than that on y, so *ceteris paribus* the student would rather have another burrito. But chips cost more than burritos, so a free basket of chips could be *exchanged* for \$3, with which 1.5 burritos can be purchased. ▲

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