

Solutions to Midterm II

Math 20
Introduction to Multivariable Calculus and Linear Algebra

April 14, 2006

Rules:

- This is a one-hour exam.
- Calculators are not allowed.
- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.
- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*
- Please do not put your name on any page besides the first page. If you like, you may put your ID number on the top of each page you write on.

Hints:

- Read the entire exam to scan for obvious typos or questions you might have.
- Budget your time so that you don't run out.
- Problems may stretch across several pages.
- Relax and do well!

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.

—*Handbook for Students*

Summary Data

Problem	1	2	3	4	5	6	Total	Percent
Max Possible	9	16	9	9	10	7	60	100%
Max Achieved	9	16	9	9	10	5	57	95%
Mean	8.64	12.92	8.17	7.92	8.31	4.67	50.61	84%
Median	9.0	14.0	9.0	9.0	9.5	5.0	53.5	89%
Mode	9	16	9	9	10	5	54	90%
% full credit	83%	23%	53%	57%	47%	0%	0%	0%
% no credit	0%	0%	3%	7%	0%	0%	0%	0%
Standard Deviation	0.7510	2.9662	1.6750	2.3139	2.0793	0.6236	6.9213	12%
Correlation with Total	0.3096	0.8265	0.4608	0.7334	0.7340	0.3883	1.0000	1.0000

1. (9 Points) Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{bmatrix}$$

Find the reduced row echelon form of A .

Solution. We have

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{bmatrix} \begin{array}{l} | \cdot (-1) \\ | \cdot (-2) \\ \leftarrow + \\ \leftarrow + \end{array} \\
 &\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & -2 & -6 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow + \\ \leftarrow + \end{array} \\
 &\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix} \begin{array}{l} \\ \\ | \cdot (-1) \\ \leftarrow + \end{array} \\
 &\rightsquigarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow + \\ \leftarrow + \\ | \cdot (-1) \\ | \cdot (-2) \end{array} \\
 &\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \leftarrow + \\ | \cdot (-1) \\ \\ \end{array} \\
 &\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$



2. (16 Points) Let A be a matrix whose reduced row echelon form is the matrix

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) In the system of equations $A\mathbf{x} = \mathbf{0}$, what are the free variables?

Solution. The free variables correspond to the non-pivot columns of A . These are the third and fifth. Hence x_3 and x_5 are pivot columns. ▲

(ii) Find the parametric form of the general solution to the system of equations $A\mathbf{x} = \mathbf{0}$.

Solution. We already know x_5 and x_3 are free. Reading off from the RREF, we have $x_4 = x_5$, $x_2 = -3x_3$, and $x_1 = -2x_3$. Thus the solution has general form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -3x_3 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

▲

(iii) Find the rank of A and the nullity of A .

Solution. The rank of A is the number of nonzero rows of R . This number is 3. Since the rank and the nullity of A add up to the number of columns of A , the nullity must be 2. ▲

(iv) Are the columns of A linearly independent? Justify your answer.

Solution. No, the columns of A are not linearly independent. The third column of R is twice the first plus three times the second, for example. ▲

(v) Do the columns of A span \mathbb{R}^4 ? Justify your answer.

Solution. The dimension of the column space of A is the rank of A , which we have shown to be 3. So the columns of A span a three-dimensional subspace, which cannot be \mathbb{R}^4 . ▲

(vi) Give a basis for the null space of A .

Solution. We showed in part (ii) that the solutions to the system $A\mathbf{x} = \mathbf{0}$ are linear combinations of $(-2, -1, 1, 0, 0)$ and $(0, 0, 0, 1, 1)$. Another way of saying that is that these two vectors span the null space of A . They are linearly independent, so they form a basis. ▲

(vii) Which columns of A form a basis for the column space of A ? (Note: we don't know what A is, but we can still answer this question)

Solution. Since the first, second, and fourth columns of R span its column space, the same columns in A span the column space of A . ▲

(viii) Give a basis for the row space of A .

Solution. The row space is spanned by the nonzero rows of R , which are $(1, 0, 2, 0, 0)$, $(0, 1, 3, 0, 0)$, and $(0, 0, 0, 1, -1)$. ▲

3. (9 Points) Let

$$A = \begin{bmatrix} 0 & 3 & -2 & 0 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 1 & 0 \end{bmatrix}$$

Find the determinant of A .

Solution. We expand by cofactors along the last column. Then we similarly choose the row or column of each subdeterminant with the most zeroes.

$$\begin{aligned} \begin{vmatrix} 0 & 3 & -2 & 0 \\ 1 & 2 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & 0 & 1 & 0 \end{vmatrix} &= - \begin{vmatrix} 0 & 3 & -2 \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 3 & -2 \\ 1 & 2 & -1 \\ 3 & 0 & 1 \end{vmatrix} \\ &= -3 \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} \\ &= -3(6+2) - 3 + 3(-3+4) = -24 \end{aligned}$$



4. (9 Points) Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$$

Find an orthonormal basis for the span of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Solution. We will first find an orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ using the Gram-Schmidt process. Then we will normalize it to an orthonormal set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

We have $\mathbf{v}_1 = \mathbf{u}_1$. Then $\mathbf{v}_1 \cdot \mathbf{v}_1 = 4$ and $\mathbf{v}_1 \cdot \mathbf{u}_2 = 4$. Thus

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{v}_1 \cdot \mathbf{u}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

Now $\mathbf{v}_2 \cdot \mathbf{v}_2 = 8$, $\mathbf{u}_3 \cdot \mathbf{v}_1 = 8$, and $\mathbf{u}_3 \cdot \mathbf{v}_2 = -4$. So

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{v}_1 \cdot \mathbf{u}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 \\ &= \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

These vectors are orthogonal, leading to the orthonormal basis

$$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \left\{ \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$



5. (10 Points) *Every morning a student wakes up and decides whether to go to class. If he went to class yesterday, he decides with probability $\frac{1}{2}$ to skip, but if he skipped yesterday, he decides with probability $\frac{3}{4}$ to go to class. We'll analyze this situation as a Markov process.*

(a) (3 points) *What is the transition matrix?*

Solution. We have

$$A = \begin{bmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{bmatrix}$$



(b) (3 points) *If the student goes to class on Monday, what's the probability he will go to class on Wednesday (two days later)?*

Solution. The transition over *two* days is the square of the transition matrix.

$$A^2 = \begin{bmatrix} 5/8 & 9/16 \\ 3/8 & 7/16 \end{bmatrix}$$

So the probability of going to class two days from now is $\frac{5}{8}$.



(c) (4 points) *Approximately what portion of class will the student attend?*

Solution. We are looking for the steady-state vector which satisfies $A\mathbf{x} = \mathbf{x}$ and $x_1 + x_2 = 1$. The first equation implies $(A - I)\mathbf{x} = \mathbf{0}$, so we can get it by row-reducing

$$\begin{bmatrix} -1/2 & 3/4 \\ 1/2 & -3/4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}.$$

We can actually insert the second equation into this matrix, although then we can't ignore the vector of right-hand sides:

$$\begin{bmatrix} 1 & -3/2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3/5 \\ 0 & 1 & 2/5 \end{bmatrix}.$$

Therefore the student will go to class $\frac{3}{5}$ of the time.



6. (7 Points) *Two unrelated quickies. Make sure you justify your answers.*

I. (3 points) *Suppose A is a square matrix whose reduced row echelon form has a row of zeros. Is A invertible?*

Solution. If the RREF of A has a row of zeroes, the rank of A must be less than the number of rows. So the matrix cannot be invertible. ▲

II. (4 points) *Suppose S is a finite subset of \mathbb{R}^n . Under what conditions is S a subspace of \mathbb{R}^n ?*

Solution. If \mathbf{v} is any nonzero vector in S , and S is a subspace of \mathbb{R}^n , then S must contain *any* multiple of \mathbf{v} . Hence if S contains any nonzero vector and is a subspace, it must be infinite. Therefore the only finite subspace of \mathbb{R}^n is the set $\{\mathbf{0}\}$. ▲