

Name: \_\_\_\_\_ ID#: \_\_\_\_\_

## Solutions to Midterm III

Math 20  
Introduction to Multivariable Calculus and Linear Algebra

May 5, 2006

Rules:

- This is a one-hour exam.
- Calculators are not allowed.
- Unless otherwise stated, show all of your work. Full credit may not be given for an answer alone.
- You may use the backs of the pages or the extra pages for scratch work. *Do not unstaple or remove pages as they can be lost in the grading process.*
- Please do not put your name on any page besides the first page. If you like, you may put your ID number on the top of each page you write on.

Hints:

- Read the entire exam to scan for obvious typos or questions you might have.
- Budget your time so that you don't run out.
- Problems may stretch across several pages.
- Relax and do well!

*Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.*

—*Handbook for Students*

## Summary Data

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6	Total	Percent
Maximum Possible	13	10	8	9	10	10	60	100.00%
Maximum Achieved	13	10	8	9	10	10	59	98.33%
Mean	10.81	9.78	3.58	8.42	7.67	3.69	43.94	73.24%
Median	12	10	4	9	8	3	44.5	74.17%
Mode	13	10	4	9	7	2	43	71.67%
% full credit	23%	80%	10%	77%	10%	0%	0%	0.00%
% no credit	0%	0%	13%	0%	0%	3%	0%	0.00%
Standard Deviation	2.6228	0.4779	2.4195	1.2775	2.1985	2.5584	6.6789	11.13%
Correlation with Total	0.6813	0.0571	0.5332	0.3120	0.7063	0.6346	1.0000	100.00%

1. (13 Points)

(i) (3 points) Let

$$A = \begin{bmatrix} 10 & -4 \\ 24 & -10 \end{bmatrix}$$

Find the eigenvalues of  $A$ .

*Solution.* The characteristic polynomial of  $A$  is

$$(10 - \lambda)(-10 - \lambda) + 24(4) = \lambda^2 - 4,$$

so the eigenvalues are  $\pm 2$ . ▲

(ii) (3 points) Let

$$B = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 4 & 3 \\ 0 & -6 & -5 \end{bmatrix}$$

Is 1 an eigenvalue of  $B$ ?

*Solution.* 1 is an eigenvalue of  $B$  if  $B - I$  is not invertible. This would be indicated by a row of zeros or a non-pivot column in the reduced row echelon form of  $B - I$ .

$$B - I = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 3 \\ 0 & -6 & -6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So 1 is an eigenvalue of  $B$ . Alternatively, one may find the characteristic polynomial of  $B$ , which is  $-\lambda^3 + \lambda^2 + 4\lambda - 4$ . Plugging in 1 to this polynomial gives zero, so 1 is an eigenvalue of  $B$ . ▲

Finding the characteristic polynomial is the harder of the two ways to do this problem. Many mistakes were made along this route because of arithmetic errors in the determinant. Also, once you have the characteristic polynomial, testing that 1 is a root by *plugging it in* is much easier than factoring the polynomial and seeing if  $(\lambda - 1)$  is a factor.

(iii) (7 points) Let

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -3 & -2 \\ 0 & 4 & 3 \end{bmatrix}$$

The eigenvalues of  $C$  are 1 and  $-1$ . Is  $C$  diagonalizable?

*Note.* It is not enough to note that there are fewer eigenvalues than columns.

*Solution.* We need to make sure we have enough eigenvectors. Since we have fewer than three eigenvalues, we will have to find an eigenvalue with at least two linearly independent eigenvectors associated to it. We have

$$C - I = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus  $x_1$  is free and  $x_2 = x_3 = 0$ . So  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is an eigenvector for  $C$  associated to the eigenvalue 1. No other linearly independent eigenvector exists; we would need another free variable.

On the other hand,

$$C + I = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 4 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here  $x_3$  is the free variable,  $x_2 = -x_3$  and  $x_1 = 0$ . So  $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$  is an eigenvector for  $C$  associated to the eigenvalues  $-1$ .

There are only two linearly independent eigenvectors for  $C$ , and we would need three to diagonalize  $C$ . Therefore,  $C$  is not diagonalizable. ▲

Another way to do it would be to look at the characteristic polynomial of  $C$ , which is

$$-\lambda^3 + \lambda^2 + \lambda - 1 = -(\lambda - 1)^2(\lambda + 1)$$

Since the eigenvalue 1 is repeated we know that this is the source of the possible trouble. Then we can find as above that the eigenspace for  $\lambda = 1$  is one-dimensional, which is too small.

2. (10 Points) Consider the following linear programming problem: Maximize

$$z = 3x_1 + x_2$$

subject to constraints

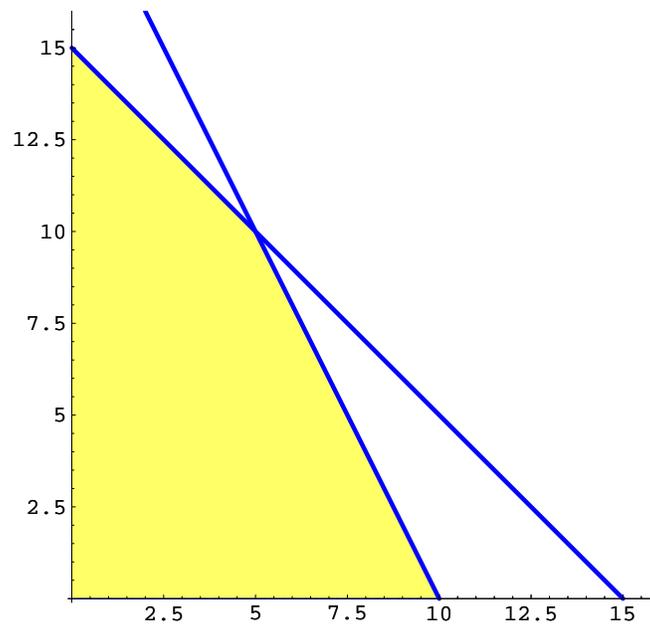
$$x_1 + x_2 \leq 15$$

$$2x_1 + x_2 \leq 20$$

and  $x_1 \geq 0, x_2 \geq 0$ .

(a) (7 points) Draw the feasible set and label all corners.

*Solution.* The line  $x_1 + x_2 = 15$  has intercepts  $(15, 0)$  and  $(0, 15)$ . The line  $2x_1 + x_2 = 20$  has intercepts  $(10, 0)$  and  $(0, 20)$ . The feasible set is “under” both of these lines. The intersection is at  $(5, 10)$ .



(b) (3 points) Solve the problem.

*Solution.* We only need to evaluate the function at each of the corner points:

$$z(5, 10) = 20$$

$$z(0, 0) = 0$$

$$z(10, 0) = 30$$

$$z(0, 15) = 15$$

Apparently the maximum value is 30, achieved at  $(10, 0)$ .



3. (8 Points) A paper mill produces rolls of paper each of which is 18 ft. wide. These are then cut into various widths as required by customers. A roll can be cut into narrower rolls as many times as necessary.

A customer requires rolls in widths of 9 ft., 7 ft. and 5 ft. There are six ways to do this:

- Cut two rolls 9 ft. wide
- Cut one 9 ft. and one 7 ft., wasting 2 ft.
- Cut one 9 ft. and one 5 ft., wasting 4 ft.
- Cut two 7 ft., wasting 4 ft.
- Cut one 7 ft. and two 5 ft., wasting 1 ft.
- Cut three 5 ft., wasting 3 ft.

(a) (5 points) An order is received for 10 rolls 9 ft. wide, 20 rolls 7 ft. wide, and 50 rolls 5 ft. wide. Formulate a linear programming problem to fill the order using the minimum number of rolls. Do not solve the problem!

*Solution.* We have a decision variable for each “type” of roll cut. Call them  $y_1, \dots, y_6$ . The number of rolls cut is the quantity

$$w = y_1 + y_2 + y_3 + y_4 + y_5 + y_6$$

which we want to minimize. We have to fill the order, though, so we have to produce a minimum number of 9, 7, or 5 foot rolls. Thus we have constraints:

$$\begin{array}{rcl} 2y_1 + y_2 + y_3 & & \geq 10 \\ y_2 & + 2y_4 + y_5 & \geq 20 \\ y_3 & & + 2y_5 + 3y_6 \geq 50 \end{array}$$



(b) (3 points) Formulate the dual problem to this problem. Don't solve it either.

*Solution.* We need to reverse the objective and the constraint inequalities, transpose the coefficient matrix, and exchange the coefficients of the objectives with the right-hand sides of the constraints. Thus we must *maximize* the quantity

$$z = 10x_1 + 20x_2 + 50x_3$$

subject to constraints

$$\begin{array}{rcl} 2x_1 & & \leq 1 \\ x_1 + x_2 & & \leq 1 \\ x_1 & + & x_3 \leq 1 \\ 2x_2 & & \leq 1 \\ x_2 + 2x_3 & & \leq 1 \\ & & 3x_3 \leq 1 \end{array}$$

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This we could use our simplex method on. ▲

This problem turned out to be pretty hard. It was probably the multiple uses of the word “roll” as a product and an ingredient of sorts. I tried to grade generously, giving partial credit for any linear programming problem in (a) and any demonstration that you knew what the dual problem was. Stabs at a solution to (a) were often close to the dual problem. The average score was 4/8 on this problem.

4. (9 Points) Consider a standard linear program whose initial tableau is

	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$	$z$	value
$y_1$	2	1	1	0	0	0	20
$y_2$	1	1	0	1	0	0	15
$y_3$	1	0	0	0	1	0	8
$z$	-3	-1	0	0	0	1	0

(a) (3 points) What is the next entering variable? The departing variable?

*Solution.* To find the entering variable, we look for the largest negative entry in the objective ( $z$ ) row. Apparently this is in the first column, so  $x_1$  is the entering variable.

We compute the  $\theta$ -ratios, which are the ratios of the last column to the columns of the new entering variable. We have  $\frac{20}{2} = 10$ ,  $\frac{15}{1} = 15$ , and  $\frac{8}{1} = 8$ . The row with the lowest positive  $\theta$ -ratio is the departing variable. This is  $y_3$ .



The most common mistake (which wasn't that common) was to misidentify the departing variable. If the wrong departing variable is chosen, the new basic solution will violate some of the constraints.

(b) (6 points) Write down the next tableau.

*Solution.* We use row operations to zero out the  $x_1$  column except in the  $x_1$  row (not the first row, but the row which  $x_1$  is in, which happens to be the third). That means, we subtract 2 times row 3 from row 1, subtract row 3 from row 2, and add 3 times row 3 to row 4. We get a new tableau:

	$x_1$	$x_2$	$y_1$	$y_2$	$y_3$	$z$	value
$y_1$	0	1	1	0	-2	0	4
$y_2$	0	1	0	1	-1	0	7
$x_1$	1	0	0	0	1	0	8
$z$	0	-1	0	0	3	1	24



## 5. (10 Points)

I. Consider a game with payoff matrix

$$\begin{bmatrix} 0 & -1 & 7 \\ 2 & 3 & 8 \\ 9 & 5 & 6 \\ 10 & 4 & -2 \end{bmatrix}$$

What are the optimal strategy or strategies for the row and column player?

*Solution.* This matrix has a saddle point in the (3,2) position. Thus  $R$  should choose strategy 3 and  $C$  should choose strategy 2. The payoff is 5. ▲

Another possible solution would be to use the simplex method, but the procedure is quite drawn out. You must add 3 to every element of the matrix to ensure the value is positive, then set up the problem, solve the problem, then divide by the value to get the percentages for each strategy. That's why I put in the saddle point!

II. A student is studying for an exam. She assumes the exam will contain a mixture of conceptual and computational problems. She estimates that if she spends all her time studying conceptual problems, she will be able to do 80% of the conceptual problems and 70% of the computational problems on the exam. On the other hand, if she studies only computational problems, she will be able to do 90% of the computational problems but only 50% of the conceptual problems on the exam.

The student has 15 hours to study for the exam. How should she budget her time? If her estimates are correct, what is her minimum score on the exam?

*Solution.* The student is playing a game against the test (or maybe the author of the test). The payoff (percentage of problems correct) has a matrix like this:

		Exam has	
		Conceptual	Computational
Student studies	Conceptual	0.8	0.7
	Computational	0.5	0.9

This is a  $2 \times 2$  game with no saddle point and there is a simple formula: The percentage devoted to studying conceptual problems should be

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{22}} = \frac{0.4}{0.5} = 80\%.$$

This means the student should study conceptual problems for 12 hours and computational problems for 3 hours. The minimum payoff, that is, the minimum percentage correct, is

$$\frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - a_{12} - a_{22}} = \frac{0.37}{0.5} = 74\%.$$



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The point of game theory is that score (or higher) is guaranteed no matter what the content of the exam is. If the professor is trying to write the most difficult exam possible, he should make

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{0.2}{0.5} = 40\%$$

conceptual problems and 60% computational problems. But this assumes the professor's objective is to minimize the students's scores, which sounds a little diabolical to me!

6. (10 Points) Let  $A$  be a symmetric  $2 \times 2$  matrix.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

(a) Show that  $A$  has real eigenvalues.

*Note.* Here  $a$ ,  $b$ , and  $c$  are arbitrary real numbers. This does not mean give them arbitrary values and show that that matrix has real eigenvalues. Leave  $a$ ,  $b$ , and  $c$  unknown but still show how the eigenvalues cannot have any imaginary parts to them.

*Solution.* The characteristic polynomial of  $A$  is

$$(a - \lambda)(b - \lambda) - b^2 = \lambda^2 - (a + c)\lambda + ac - b^2.$$

So the eigenvalues are

$$\lambda = \frac{a + c \pm \sqrt{(a + c)^2 - 4(ac - b^2)}}{2} = \frac{a + c \pm \sqrt{(a - c)^2 + 4b^2}}{2}.$$

The expression in the radical is nonnegative, so the eigenvalues are always real. ▲

(b) Show that  $A$  is either diagonalizable or already diagonal.

*Solution.* If the expression under the radical is positive, there are two distinct real eigenvalues, and so the matrix is diagonalizable. The only other possibility is that the eigenvalues repeat, that is, the expression under the radical is zero. This would mean that  $a = c$  and  $b = 0$ . Thus  $A$  is of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}. \quad \text{▲}$$