Lesson 27
Linear Programming; The Simplex Method

Math 20

April 19, 2006

1 Setup

A **standard linear programming problem** is to maximize the quantity

$$c_1x_1 + c_2x_2 + \dots c_nx_n = \mathbf{c}^\mathsf{T}\mathbf{x}$$

subject to constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\le b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\le b_m \end{aligned}$$

or

$$A\mathbf{x} \le \mathbf{b}.$$

We usually include the nonnegativity constraint $\mathbf{x} \ge \mathbf{0}$. Today we will also assume $\mathbf{b} \ge 0$.

Any vector $\mathbf{x}$ which satifies all the inequalities is called a **feasible solution** to the given problem, and a feasible solution maximizing the objection function is called an **optimal solution**.

2 Illustrative Problem

We will use the baker of before. He is trying to maximize

$$z = 8x + 10y$$

subject to the constraints

$$\begin{aligned} 2x + y &\le 50 \\ x + 2y &\le 70 \\ x &\ge 0 \\ y &\ge 0. \end{aligned}$$

3 Slack Variables

We can turn the inequalities into equalities by inserting new variables, which are called *slack variables*. Thus the first equation of constraint becomes

$$2x + y + u = 50,$$

and the second

$$x + 2y + v = 70.$$

But u and v are nonnegative. So the new problem is to maximize $8x + 10y$ subject to constraints

$$\begin{aligned} 2x + \ y + u \quad &= 50 \\ x + 2y \quad + v &= 70 \end{aligned}$$

$$x \geq 0, \qquad y \geq 0, \qquad u \geq 0, \qquad v \geq 0.$$

In general, we insert slack variables $u_1, u_2, \ldots, u_m$ and the equations of constraint become

$$A\mathbf{x} + \mathbf{u} = \mathbf{b},$$

along with $\mathbf{x} \geq \mathbf{0}$, $\mathbf{u} \geq \mathbf{0}$.

Definition. The vector $\mathbf{x}$ in $\mathbb{R}^{n+m}$ is called a **basic solution** if its obtained by setting n of the variables in this equation equal to zero and solving for the remaining n variables. The m variables are we solve for are called the **basic variables**, and then n variables set equal to zero are called the **nonbasic variables**. The vector $\mathbf{x}$ is called a **basic feasible solution** if it is a basic solution that also satisfies the inequalities $\mathbf{x} \geq \mathbf{0}$.

Why are basic feasible solutions necessary?

Theorem. *If a LP problem has an optimal solution, then it has a basic optimal solution.*

So we only need to find the basic feasible solutions!

How many basic feasible solutions are there? Out of the $m+n$ variables, we choose n to set equal to zero, and solve for the rest.[1]. This can be done

$$\binom{n+m}{m} = \frac{(n+m)!}{m!n!}$$

ways. That's a lot!

The **simplex method** is a way to arrive at an optimal solution by traversing the vertices of the feasible set, in each step increasing the objective function by as much as possible.

[1] I believe that a nonsolvability here would come from linearly dependent constraints, some of which could be eliminated

4 The Simplex Method, By Example

4.1 The Initial Basic Feasible Solution

We'll work with the illustrative problem. We can start with the basic feasible solution $x = 0$, $y = 0$. This means $u = 50$ and $v = 70$.

We'll start writing everything in a table (or **tableau**), so let's also write the objective function with a right-hand side of zero. Thus

$$-8x - 10y + z = 0.$$

We put this all together, forming what is called the **initial tableau**:

	x	y	u	v	z	value
u	2	1	1	0	0	50
v	1	2	0	1	0	70
	-8	-10	0	0	1	0

Looking at the last row, we can see that this is not the optimal solution. Increasing x or y would result in an increase in z to balance out. The idea in general is the:

Optimality Criterion. *If the objective row of a tableau has no negative entries in the columns labeled with variables, then the indicated solution is optimal and we can stop our computation.*

The idea is this: We're going to move from one basic solution to another. That is, we need to make one of the nonzero (basic) variables zero and one of the zero (nonbasic) variables nonzero. We want to do this as efficiently as possible.

Look at the last row. The -10 in the y column indicates that an increase in y would have to be balanced by a tenfold increase in z. This multiplier is the biggest one. So we choose to make this variable nonzero (basic). Since it's entering the set of basic variables, it's called the **entering variable**.

How much can we increase y? Well, since x is still zero, the equations of constraint can be written

$$\begin{aligned} u &= 50 - y \\ v &= 70 - 2y. \end{aligned}$$

We still need $u \geq 0$ and $v \geq 0$, so the most we can increase y is to 35. This is the smallest of the ratios $\frac{50}{1} = 50$ and $\frac{70}{2} = 35$. So we're going to increase y to 35. This will make $v = 0$. We call v the **departing variable**.

The new basic solution therefore has $y = 35$, $v = 0$, $u = 15$, and $x = 0$. The new value of the objective function is $z = 10y = 350$.

4.2 Creating a New Tableau

We are exchanging the basic variable v for y. This means the objective row has to be replaced with one that has a zero in the y columns. We can do this by adding multiples of row 2. Since y is an entering variable, we might as well normalize row 2 to have a one.

So we scale the second row to have a one in the y column

x	y	u	v	z	value
2	1	1	0	0	50
$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	35
-8	-10	0	0	1	0

Now we zero out the rest of this column by adding 10 times row 2 to row 3, and subtracting row 2 from row 1.

	x	y	u	v	z	value
u	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	0	15
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	35
	-3	0	0	5	1	350

Now we can repeat the process! The x column in the objective row has a negative entry, so increasing x will increase z. How much can we increase it? The minimum of the two ratios $\frac{15}{3/2} = 10$ and $\frac{35}{1/2} = 70$. So x in the entering variable and u is the departing variable.

We scale row one by $\frac{2}{3}$ to make it one in the basic column:

	x	y	u	v	z	value
u	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	10
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	35
	-3	0	0	5	1	350

And we zero out the rest of the column by subtracting half of row 1 from row 2, and adding 3 times row 1 to row 3.

	x	y	u	v	z	value
u	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	10
y	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	30
	0	0	2	4	1	380

Now any increase in the decision variables or slack variables would result in a *decrease* of z. We are done!

5 Recap of Steps

1. Set up the initial problem.

2. Apply the optimality test. If the objective row has no negative entries in the columns labeled with variables, then the indicated solution is optimal; we can stop.

3. Choose a pivotal column by determining the column with the most negative entry in the objective row. If there are several candidates for a pivotal column, choose any one.

4. Choose a pivotal row. Form the ratios of the entries above the objective row in the rightmost column by the corresponding entries of the pivotal column for those entries in the pivotal column which are positive. For some reason, these ratios are called ***θ*-ratios** (theta). The pivotal row is the row for which the smallest of these ratios occurs. If there is a tie, so that the smallest ratio occurs at more than one row, choose any one of the qualifying rows. If none of the entries in the pivotal column above the objective row is positive, the problem has no finite optimum. We stop.

5. Perform pivotal elimination to construct a new tableau and return to Step 2.

6 Example

Let's maximize $z = 3x_1 - x_2 + 6x_3$ subject to the constraints

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &\leq 4 \\ -2x_1 + 2x_2 - 3x_3 &\geq -4 \\ 2x_1 + x_2 - x_3 &\leq 8 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0. \end{aligned}$$

Negating row two puts this problem into standard form.

Let's do this with the simplex method. We insert slack variables u_1, u_2, and u_3 The equations of constraint become

$$\begin{aligned} 2x_1 + 4x_2 + x_3 + u_1 \qquad\qquad &\leq 4 \\ 2x_1 - 2x_2 + 3x_3 \qquad + u_2 \qquad &\leq 4 \\ 2x_1 + x_2 - x_3 \qquad\qquad + u_3 &\leq 8 \end{aligned}$$

with all variables nonnegative.

An initial basic feasible solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $u_1 = 4$, $u_2 = 4$, $u_3 = 8$. The initial tableau is therefore

	x_1	x_2	x_3	u_1	u_2	u_3	z	value
u_1	2	4	1	1	0	0	0	4
u_2	3	-2	3	0	1	0	0	4
u_3	2	1	-1	0	0	1	0	8
z	-3	1	-6	0	0	0	1	0

The entering variable is the one with the largest negative coefficient in the objective row, that is, x_3. The departing variable is the row where the last column divided by the third column has its minimum positive value. These θ-ratios are $\frac{4}{3}$ and $\frac{4}{1}$. Apparently, then u_2 is the departing variable. We scale row 2 by $\frac{1}{3}$, and use row operations to zero out the rest of the column. We get

	x_1	x_2	x_3	u_1	u_2	u_3	z	value
u_1	1	$\frac{14}{3}$	0	1	$-\frac{1}{3}$	0	0	$\frac{8}{3}$
x_3	1	$-\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{4}{3}$
u_3	3	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1	0	$\frac{28}{3}$
z	3	-3	0	0	2	0	1	8

Now the entering variable is x_2. The θ-ratios are

$$\frac{8/3}{14/3} = \frac{3}{14}$$
$$\frac{28/3}{1/3} = 28.$$

So the departing variable is u_1. We scale row 1 by $\frac{3}{14}$ and use row operations to zero out the rest. We get

	x_1	x_2	x_3	u_1	u_2	u_3	z	value
x_2	$\frac{3}{14}$	1	0	$\frac{3}{14}$	$-\frac{1}{14}$	0	0	$\frac{4}{7}$
x_3	$\frac{8}{7}$	0	1	$\frac{1}{7}$	$\frac{2}{7}$	0	0	$\frac{12}{7}$
u_3	$\frac{41}{14}$	0	0	$-\frac{1}{14}$	$\frac{5}{14}$	1	0	$\frac{64}{7}$
z	$\frac{51}{14}$	0	0	$\frac{9}{14}$	$\frac{25}{14}$	0	1	$\frac{68}{7}$

Now there is nothing else to increase; we have reached the optimum value $\frac{68}{7}$ at $x_1 = 0$, $x_2 = \frac{4}{7}$, and $x_3 = \frac{12}{7}$.