Problem set 5

Math 212a

October 19, 2000, Due October 31

Contents

1 Three standard counterexamples.

1.1 Non-dominated convergence.

Let $f_n := n\mathbf{1}_{(0,\frac{1}{n}]}$. This was one of the examples we studied in class where $\int_{\mathbf{R}} f_n dx \equiv 1$ while $\lim f_n(x) = 0$ for all x. So we get strict inequality in Fatou's lemma, and can not move the limit past the integral sign. So we know in advance that this can not be a case of dominated convergence.

1. Let g be defined by $g(x) := \sup_k f_k(x)$. We know that $g \notin \mathcal{L}_1$. But compute g and show explicitly that its integral diverges.

1.2 Order of integration matters for non-integrable functions.

2. Take $X = Y = [0, 1]$ with standard Lebesgue measure. Let

$$f(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } \ 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } \ 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases} .$$

Compute $\int_X \left(\int_Y f(x,y)dy\right) dx$ and $\int_Y \left(\int_X f(x,y)dx\right) dy$ and verify that they are not equal. This implies that $f \notin \mathcal{L}_1(X\times Y, \mathbf{R})$ for otherwise Fubini would apply. Show directly that both f^+ and f^- have infinite integrals on $X \times Y$.

1.3 Order of integration matters in the non-σ-finite case.

Let $X = Y = [0,1]$ as before. Take $\mathcal{F}$ to be the usual Lebesgue measurable sets and m to be Lebesgue measure. Take $\mathcal{G}$ to be "all" subsets, and take n to be

$$n(A) = \text{number of elements in } \ A$$

(in particular $n(A) = \infty$ if A is not a finite set). Since $[0,1]$ is not the countable union of finite sets, it is not σ-finite. So we expect that Fubini might fail. Let $\Delta \subset X \times Y$ be the diagonal, i.e.

$$\Delta := \{(x,y)|x = y\}.$$

3. Show that $\Delta \in \mathcal{F} \times \mathcal{G}$. Let

$$f := \mathbf{1}_\Delta.$$

Evaluate $\int_X \left(\int_Y f(x,y)dy\right) dx$ and $\int_Y \left(\int_X f(x,y)dx\right) dy$ and verify that they are not equal.

2 Steiner symmetrization.

J. Steiner (1796-1863) published a series of papers mainly in the years 1838-1842 devoted to the isoperimetric inequality for the plane and for three dimensional space. In these papers, he introduced various procedures called "symmetrization". His idea was that you could produce a figure with the same volume (or area in the plane) which was more symmetric than the original one and whose surface area (or perimeter in the plane) is no larger than the original one. The "rigor" of his arguments was criticized by Dirichlet and Weierstrass and the like, but his ideas have proved to be a source of much investigation and and eventually justified. Brunn, in his thesis in 1887, proved some basic inequalities for convex bodies in two and three dimensions. In a series of stunning papers published between 1897-1911 Minkowski (1864-1909) revolutionized the subject, yielding, among other things, an inequality which has the isoperimetric inequality as a very special case. The purpose of the next few problems is to give you a taste of this subject, which is not part of the standard curriculum here at Harvard.

One of the procedures known as Steiner symmetrization is the following: We start with some compact set K in $\mathbf{R}^n$ and choose some subspace H of dimension $n-1$. Each line perpendicular to H at a point $h \in H$ will intersect K in some subset K_h, which will be measurable for almost all h, and its one dimensional measure $m(K_h)$ is a bounded measurable function of h. Fubini implies that

$$\int_H m(K_h)dh = \text{vol }(K)$$

where vol (K) denotes the n-dimensional measure of the set K. Steiner symmetrization replaces each of the sets K_h by an interval of length $m(K_h)$ placed symmetrically about H along the perpendicular through h. To be more explicit, suppose that H consists of all points whose last coordinate equals 0. So we write the most general element of $\mathbf{R}^n$ as (h, y) where $h \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. Then the Steiner symmetrization of K with respect to H is

$$\text{St}_H(K) := \{(h,y)| - \frac{1}{2}m(K_h) \leq y \leq \frac{1}{2}m(K_h)\}.$$

For example, we reproduce below the Steiner symmetrization of the equilateral triangle through the line passing through one of its sides: You might ask why replace the beautiful equilateral triangle by the rather ugly parallelepiped? The reason is that the parallelepiped is contained is a circle of radius $\frac{1}{2}$ (where the side length of the triangle is 1) while the circumscribed circle for the triangle has radius $1/2\sin(\pi/3) \doteq .577...$ For any set A in a metric space we have defined its diameter and the equilateral triangle shows that A need not be contained in a ball whose radius is $\frac{1}{2}\,\text{diam}(A)$. However, without changing the volume, Steiner symmetrization has replace the equilateral triangle by a figure which *is* contained in a ball whose radius is half the diameter of the original triangle. So let us introduce the following notation:

$$\kappa_n = \text{vol}(B_1),$$

where B_1 denote the unit ball in Euclidean n-space. Explicitly,

$$\kappa_n = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$$

but we will not need this explicit formula. We will also need a symbol for the $n-1$-dimensional volume ("area") of the unit sphere, which we shall denote by ω_n. Geometry then says that the "area" of the sphere of radius r is $\omega_n r^{n-1}$ hence "polar coordinates" (and Fubini) say that

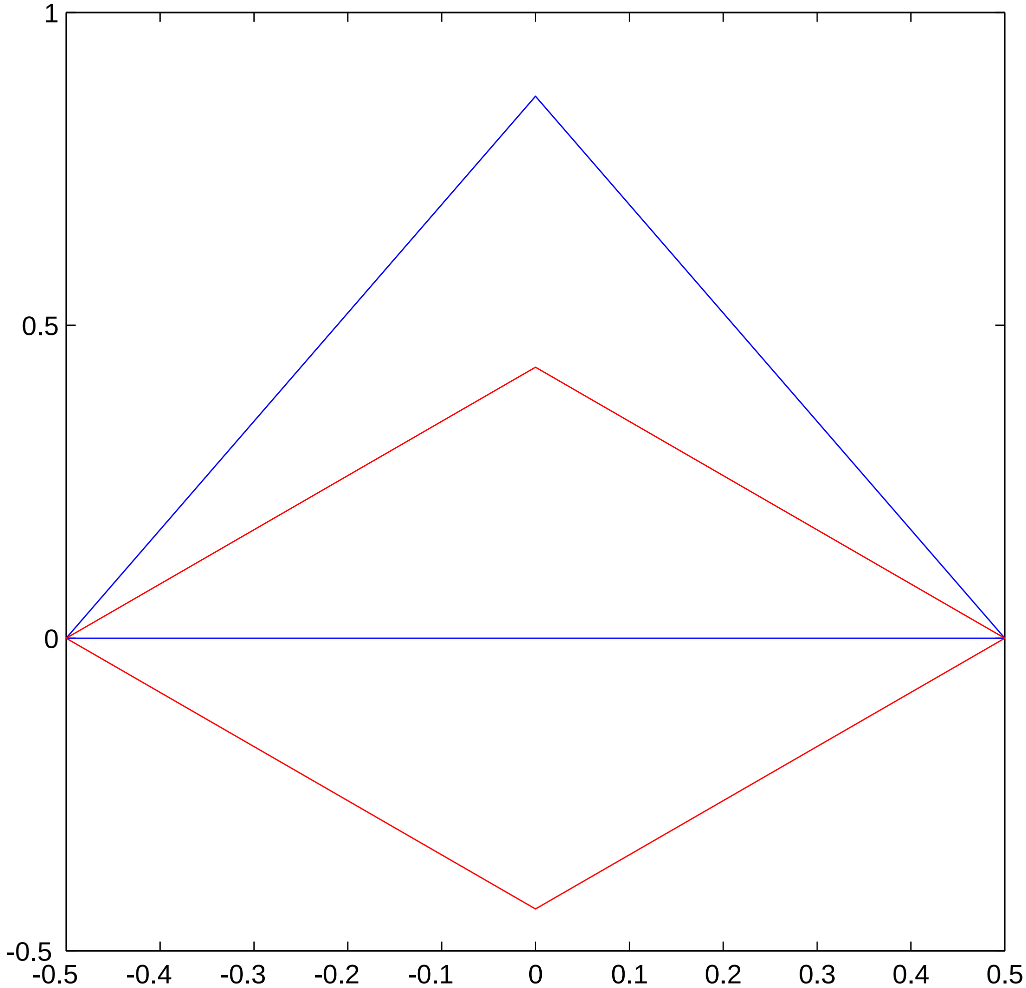
$$\kappa_n = \int_0^1 \omega_n r^{n-1} dr = \frac{1}{n}\omega_n.$$

In short

$$\omega_n = n\kappa_n.$$

(Remember this by recalling that the circumference of the unit circle is 2π which the area of the unit disk is π.) We want to prove

$$\text{vol}(A) \leq \kappa_n \left(\frac{\text{diam}(A)}{2}\right)^n \tag{1}$$



Equilateral triangle (blue) and its Steiner symmetrization (red) known as the **isodiametric inequality**.

This is a very poor inequality as can be seen by the figure. A much better inequality along the same lines is the following: Suppose that A is convex. (See below if you don't know the definition.) For each unit vector u on the unit sphere, there will be a "width of A in the u direction", i.e. maximum value of $m(A_h)$ where H is the hyperplane perpendicular to u. (For convex sets, each A_h is a (possibly empty) interval.) For each u let $\ell_A(u)$ denote the width of A in the u direction. Average this function of u over the unit sphere relative to the "area" measure of the unit sphere. The resulting number is known as the "mean width" and denoted by $b(A)$. Then **Urysohn's inequality** says that

$$\mathrm{vol}(A) \leq \kappa_n \left(\frac{b(A)}{2} \right)^n .$$

Since $\mathrm{diam}(A)$ is the maximum of the $\ell(u)$ while $b(A)$ is the average of the $\ell(u)$, it is clear that the isodiametric inequality is a consequence of (and in general weaker than) Urysohn's inequality. In fact, although we shall not prove it here, Urysohn's inequality is a very very special case of the Brunn-Minkowski theorem which we shall discuss (and assign to you to prove) in the next section.

Back to the isodiametric inequality! Here is Steiner's idea: Suppose that A is already symmetric with respect to some $n-1$ dimensional subspace L. In other words suppose that A is invariant under reflection through L. Suppose that H is perpendicular to a line in L. If we apply Steiner symmetrization to A, the resulting figure is still symmetric with respect to L. So if we apply Steiner symmetrization repeatedly, first with respect to the hyperplane perpendicular to e_1, then with respect to the hyperplane perpendicular to e_2 etc., where $e_1, \ldots, e_n$ is an orthonormal basis, we will end up with a figure (like the quadrilateral above) which is invariant with respect to transformation $x \mapsto -x$. The resulting set, call it B has the property that if $x \in B$ then $-x \in B$ so $2\|x\| = \|x-(-x)\| \leq \mathrm{diam}(B)$ in other words, B is contained in the ball of radius $\frac{1}{2}\,\mathrm{diam}(B)$ so the isodiametric inequality is obvious for B. The volume of B is the same as the volume of A. So to complete the proof of the isodiametric inequality

4. Prove that

$$\mathrm{diam}(\mathrm{St}_H(A))) \leq \mathrm{diam}(A).$$

[Hint: Any point of A can be written as $(h, y(h))$, and for a given h such that $A_h \neq \emptyset$ the maximum value of $y(h)$, call it $y^+(h)$ and the minimum value $y^-(h)$ will be achieved since we are assuming that A is compact. Steiner symmetrization replaces these two points by $(h, \pm s(h))$ where $s(h) = \frac{1}{2}m(A_h))$ so $2s(h) \leq |y^+(h) - y^-(h)|$. If $(k, y^\pm(k))$ are the two "extreme" points corresponding to another point of H, then

$$\mathrm{diam}(A) \geq \max(\|(h, y^+(h)) - (k, y^-(k)))\|, \|(h, y^-(h)) - (k, y^+(k))\|).$$

Think of Steiner symmetrization as taking place in two steps. First sliding all the sets A_h so that they become symmetric about H, and then squashing each of them down to get rid of the gaps.]

3 The Brunn-Minkowski Theorem.

A set A in $\mathbf{R}^n$ is called **convex** if $x \in A$ and $y \in A$ implies that all the points in the line segment $\{(1-t)x + ty\}, \quad 0 \le t \le 1$ belong to A. If A and B are sets, and $0 \le \lambda \le 1$, we let $(1-\lambda)A + \lambda B$ denote the set of all points $(1-\lambda)x + \lambda y, \quad x \in A, \ y \in B$. It is easy to check that if A and B are convex so are $(1-\lambda)A + \lambda B$. Suppose that $n = 1$. Then to say that A is convex means that it is an interval. If $A = [a, b]$ and $B = [c, d]$ then $(1-\lambda)A + \lambda B = [(1-\lambda)a + \lambda c, (1-\lambda)b + \lambda d]$. In one dimension volume means length and so

$$\mathrm{vol}\,((1-\lambda)A + \lambda B) = (1-\lambda)b + \lambda d - (1-\lambda)a - \lambda c = (1-\lambda)\,\mathrm{vol}(A) + \lambda\,\mathrm{vol}(B).$$

From now on A and B will denote compact convex sets in $\mathbf{R}^n$. The Brunn-Minkowski theorem asserts

Theorem 1 **[Brunn-Minkowski]** *For* $0 \le \lambda \le 1$

$$\mathrm{vol}\,((1-\lambda)A + \lambda B)^{\frac{1}{n}} \ge (1-\lambda)\,\mathrm{vol}(A)^{\frac{1}{n}} + \lambda\,\mathrm{vol}(B)^{\frac{1}{n}}.$$

Furthermore equality holds for some $0 < \lambda < 1$ *only if either* A *and* B *lie in parallel hyperplanes (in which case all volumes vanish) or* A *and* B *differ by a similarity transformation.*

We know the theorem for $n = 1$, the plan is to prove the inequality part of the theorem by induction on n.

5. Show that it is enough to prove the inequality in the Brunn-Minkowski theorem when A and B have non-empty interiors, i.e. have positive volume, and for this case it is enough to prove it for the case where both volumes $= 1$, so the inequality we wish to prove is

$$\mathrm{vol}\,((1-\lambda)A + \lambda B) \ge 1. \tag{2}$$

[Hint: Replace A by $(\mathrm{vol}(A))^{-1/n}A$ and B by $(\mathrm{vol}(B))^{-1/n}B$ and make the appropriate choice of λ in (2) to deduce the general case.]

From now on we assume that $\mathrm{vol}(A) = \mathrm{vol}(B) = 1$.

6. Let a, b, p be positive real numbers and $0 < \lambda < 1$. Show that

$$[(1-\lambda)a^p + \lambda b^p]^{\frac{1}{p}} \left[\frac{1-\lambda}{a} + \frac{\lambda}{b}\right] \ge 1. \tag{3}$$

[Hint: log is concave.]

Let $H(t)$ denote the hyperplane given by $x_n = t$. Let $A(t) := A \cap H(t)$ so $A(t)$ (possibly empty) in $n-1$ dimensions. The set of t for which the intersection is non-empty forms an interval, say $[a_A, b_A]$. Let $v_A(t)$ denote the $(n-1)$-dimensional volume of $A(t)$. So $v_A(t) > 0$ on the open interval (a_A, b_A) and by Fubini

$$\mathrm{vol}(A) = \int_{a_A}^{b_A} v_A(s)ds.$$

Let

$$w_A(t) := \int_{a_A}^{t} v_A(s)ds$$

so $w_A'(t) = v_A(t) > 0$ for $t \in (a_A, b_A)$ and w maps $[a_A, b_A]$ monotonically onto the interval $[0,1]$. Let z_A denote the inverse function. So z_A is differentiable for $0 < \tau < 1$ and

$$z_A'(\tau) = \frac{1}{v_A(\tau)} \qquad 0 < \tau < 1.$$

Write

$$k_A(\tau) := A \cap H(z(\tau)).$$

Make all the same definitions for B.

Define

$$K_\lambda := (1-\lambda)A + \lambda B$$

and

$$z_\lambda(\tau) := (1-\lambda)z_A(\tau) + \lambda z_B(\tau).$$

7. Show that

$$K_\lambda \cap H(z_\lambda(\tau)) \supset (1-\lambda)k_A + \lambda k_B(\tau)$$

and that

$$\mathrm{vol}(K_\lambda) = \int_{z_\lambda(0)}^{z_\lambda(1)} \mathrm{vol}_{n-1}(K_\lambda \cap H(t))dt = \int_0^1 \mathrm{vol}_{n-1}(K_\lambda \cap H(z_\lambda(\tau)))z_\lambda'(\tau)d\tau.$$

Here vol_{n-1} denote $n-1$ dimensional volume.

Combine these two results to conclude that

$$\mathrm{vol}(K_\lambda) \geq \int_0^1 \mathrm{vol}((1-\lambda)k_A(\tau) + \lambda k_A(v) \left[\frac{1-\lambda}{v_A(z_A(\tau))} + \frac{\lambda}{v_B(z_B(\tau))}\right] d\tau.$$

Use the full Brunn-Minkowski theorem in $(n-1)$ dimensions to conclude that this last expression is $\geq$

$$\int_0^1 [(1-\lambda)v_A^{\frac{1}{n-1}} + \lambda v_B^{\frac{1}{n-1}}]^{n-1} \left[\frac{1-\lambda}{v_A} + \frac{\lambda}{v_B}\right] d\tau$$

where we have written v_A for $v_A(z_A(\tau))$ etc. Now use Problem **5** with $p = 1/(n-1)$ to conclude the proof.

4 Mixed volumes.

I will now only sketch some of the rest of the story. For full details see the excellent book *Convex Bodies: The Brunn-Minkowski Theory* by Rolf Schneider. Let $\mathcal{K}^n$ denote the set of non-empty compact convex sets in $\mathbf{R}^n$. So if $\lambda_1, \ldots, \lambda_\ell$ are non-negative numbers, we can form the convex body = non-empty convex compact set

$$\lambda_1 K_1 + \cdots \lambda_\ell K_\ell$$

from the convex bodies $K_1, \ldots K_\ell$.

Theorem 2 [**Minkowski**] *There exists a non-negative symmetric function* $V : (\mathcal{K}^n)^n \to \mathbf{R}$ *called the* **mixed volume** *such that*

$$\mathrm{vol}\,(\lambda_1 K_1 + \cdots \lambda_\ell K_\ell) = \sum_{i_1,\ldots,i_n=1}^{\ell} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n})$$

for arbitrary convex bodies $K_1, \ldots, K_\ell$ *and arbitrary non-negative numbers* $\lambda_1, \ldots, \lambda_\ell$.

Furthermore, the mixed volume is given by

$$V(K_1, \ldots, K_n) := \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{i_1 < i_2 \cdots < i_k} \mathrm{vol}(K_{i_1} + \cdots + K_{i_k}) \qquad (4)$$

For example, if there are only two convex bodies A and B, and there are only two non-negative numbers λ and $1-\lambda$, then Minkowski's theorem asserts (using the notation used above) that

$$\mathrm{vol}(K_\lambda) = \sum_{i=0}^{n} \binom{n}{i} (1-\lambda)^{n-i} \lambda^i V_{(i)}$$

where

$$V_{(i)} := V(A, \ldots A, B, \ldots B), \quad (n-i)\ A's \ \text{ and } \ i \quad B's.$$

The Brunn-Minkowski theorem asserts that $\mathrm{vol}(K_\lambda)^{1/n}$ is a concave function of λ. Hence, subtracting off the right hand side which is linear, we conclude that

$$f(\lambda) := \mathrm{vol}(K_\lambda)^{1/n} - (1-\lambda)\,\mathrm{vol}(A)^{1/n} - \lambda\,\mathrm{vol}(B)^{1/n}$$

is a concave function of λ with $f(0) = f(1) = 0$. Hence $f'(0) \geq 0$.

8. Compute the derivative of f at 0 and conclude that

$$V(A, \ldots, A, B)^n \geq \mathrm{vol}(A)^{n-1}\,\mathrm{vol}(B). \qquad (5)$$

To understand the meaning of this profound inequality, consider the case where A is an arbitrary convex body and B is the unit ball. Then

$$A + \epsilon B$$

consists of the body obtained form A by going out a distance ϵ (perpendicular to the boundary at all smooth boundary points). Hence (this requires a little proof)

$$\frac{d}{d\epsilon}\operatorname{vol}(A+\epsilon B)_{|\epsilon=0} = S(A),$$

the surface area of the boundary. On the other hand, from the Minkowski mixed volume formula, the coefficient of ϵ is $nV(A,\dots,A,B)$. Hence, after rearranging the factors of n we see that the isoperimetric inequality

$$\left(\frac{S(A)}{\omega_n}\right)^n \geq \left(\frac{\operatorname{vol}(A)}{\kappa_n}\right)^{n-1}$$

is a special case of (5).

How about the proof of Minkowski's theorem? For details see Schneider Chapter V, but here is a sketch. A **polytope** is defined as the convex hull of finitely many points. The **facets** of a polytope are defined to be its $n-1$ dimensional faces. Each facet has an outward pointing unit vector, and the list of all these unit vectors is called the **type** of the polytope. For example, all rectangles in the plane with sides parallel to the coordinate axes are of the same type. It is not hard to prove that each convex body in a finite collection of such can be approximated (in the Hausdorff metric say) as closely as possible by polytopes all of the same type. (Of course the type gets more and more complicated as we want to make the approximations more and more close.) So to prove Minkowski's theorem, it is enough to prove it for the case where all the bodies involved are polytopes all of the same type.

Now a polytope of a given fixed type is specified by the location of its faces. So if u_i is a given unit vector in the collection of unit vectors specifying the type of the polytope P, we may let

$$h_i = \max_{y\in P}(y,u_i).$$

Then the corresponding face lies in the hyperplane given by the equation

$$(x,u_i) = h_i.$$

For polytopes P of a given type, it is not hard to prove, by induction on the dimension, that $\operatorname{vol} P$ is a homogeneous polynomial of degree n in these values h_i.

For polytopes of the same type,

$$h_i(\lambda_1 P_1 + \cdots \lambda_\ell h_i(P_\ell) = \lambda_1 h_i(P_1) + \cdots + \lambda_n(P_\ell)$$

almost by definition.

This implies that there is some expansion of $\operatorname{vol}(\lambda_1 P_1 + \cdots + \lambda_\ell P_\ell)$ into a homogeneous polynomial of degree n in the λ_i:

$$\operatorname{vol}(\lambda_1 P_1 + \cdots + \lambda_\ell P_\ell) = \sum_{i_1,\dots i_n=1}^{\ell} \lambda_{i_1}\cdots\lambda_{i_n} V(P_{i_1},\dots,P_{i_n}, \qquad (6)$$

and we will be done if we can prove that these coefficients are given by Minkowski's formula (4) with K replaced by P. This is an inclusion exclusion style argument: Define

$$f(P_1, \ldots, P_n) := \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n+k} \sum_{i_1 < i_2 \cdots < i_k} \mathrm{vol}(P_{i_1} + \cdots + P_{i_k})$$

where the P's are polytopes of the same type (including possible some $P_i = \{0\}$). We know that

$$f(\lambda_1 P_1, \ldots, \lambda_n P_n)$$

is a homogeneous polynomial of degree n in the λ_i. Now

$$(-1)^{n+1} n! f(\{0\}, P_2, \ldots, P_n) =$$

$$\sum_{2 \le i \le n} \mathrm{vol}(P_i) - \left[\sum_{2 \le j \le n} V(\{0\} + P_j) + \sum_{2 \le i < j \le n} \mathrm{vol}(P_i + P_j) \right]$$

$$+ \left[\sum_{2 \le j \le k} \mathrm{vol}(\{0\} + P_j + P_k) + \sum_{2 \le i < j < k \le n} \mathrm{vol}(P_i + P_j + P_k) \right] - \cdots$$

$$= 0.$$

So

$$f(0 \cdot P_1, \lambda_2 P_2, \ldots, \lambda_n P_n) \equiv 0.$$

This means that in the expansion of $f(\lambda_1 P_1, \ldots, \lambda_n P_n)$ into a homogeneous polynomial in the λ's, the coefficient of any monomial not involving λ_1 must vanish. But there is nothing special about the number 1. So the only surviving monomial when we expand f using (6) is the coefficient $V(P_1, \ldots, P_n)$. This establishes Minkowski's formula. QED