Problem set 7

Math 212a

November 9, 2000 due Nov. 21

The purpose of this exercise set, which is mainly computational, is to give you some experience with the ideas associated to **generalized functions**.

Recall that in our study of the Fourier transform we introduced the space $\mathcal{S}$ consisting of all functions on $\mathbf{R}$ which have derivatives of all order which vanish rapidly at infinity, and that on this space we have a countable family of norms

$$\|f\|_{m,n} := \|x^m D^n f\|_\infty.$$

Convergence in $\mathcal{S}$ means convergence with respect to all of these norms. So a linear function ℓ on this space is continuous if $|\ell(f_k)| \to 0$ whenever $\|f_k\|_{m,n} \to 0$ for all m and n. We let $\mathcal{S}'$ denote this dual space, the space of continuous linear functions on $\mathcal{S}$.

For example, consider the "Heaviside function"

$$H(x) := \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \leq 0 \end{cases}.$$

This function does not belong to $\mathcal{S}$, but it defines a continuous linear function on $\mathcal{S}$ by

$$\langle H, f \rangle = \int_{-\infty}^{\infty} f(x)H(x)dx = \int_0^{\infty} f(x)dx.$$

Of course, any element of $\mathcal{S}$ certainly defines a linear function on $\mathcal{S}$ by the same procedure, the linear function associated to $g \in \mathcal{S}$ is just the Hilbert space scalar product

$$f \mapsto (f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

This also works for any $g \in L_2(\mathbf{R})$, but $H \notin L_2(\mathbf{R})$. From the Riesz representation theorem we thus know that the linear function given by H is not continuous with respect to the L_2 norm, but it is continuous relative to the topology given above by the countable family of norms $\|\ \|_{m,n}$.

Contents

1 Differentiation of generalized functions.

The operation of differentiation

$$\frac{d}{dx} : \mathcal{S} \to \mathcal{S}, \quad f \mapsto f' = \frac{d}{dx} f$$

is a continuous linear operator. Hence it has a well defined transpose

$$\left(\frac{d}{dx}\right)^* : \mathcal{S}' \to \mathcal{S}' \qquad \left\langle \left(\frac{d}{dx}\right)^* \ell, f \right\rangle := \left\langle \ell, \left(\frac{d}{dx}\right) f \right\rangle.$$

For example, if we consider the linear function associated to an element g of $\mathcal{S}$, transpose is given by

$$f \mapsto (f', g) = \int f'(x)\overline{g(x)}dx = -\int f(x)\overline{g'(x)}dx = -(f, g')$$

by integration by parts. Thus the transpose of $\left(\frac{d}{dx}\right)$ when applied to the linear function associated to $g \in \mathcal{S}$ is the linear function associated to $-g' \in \mathcal{S}$. This suggests making the following definition: We **define**

$$\left(\frac{d}{dx}\right) : \mathcal{S}' \to \mathcal{S}'$$

by

$$\left(\frac{d}{dx}\right) := -\left(\frac{d}{dx}\right)^*.$$

For example,

$$\left\langle \left(\frac{d}{dx}\right) H, f \right\rangle = -\langle H, f' \rangle = -\int_0^\infty f'(x)dx = f(0).$$

So if we define the **Dirac delta "function"** $\delta = \delta_0 \in \mathcal{S}'$ by

$$\langle \delta, f \rangle := f(0)$$

We have

$$\frac{d}{dx} H = \delta. \tag{1}$$

We have learned how to differentiate discontinuous functions! When Heaviside wrote down formulas like this at the beginning of the twentieth century he was derided by mathematicians. When Dirac did the same, mathematicians were puzzled, but a bit more respectful because of Dirac's fantastic achievements in physics. The advent of theorems like the Riesz representation theorem gave rise to the idea that we might want to think of a function as a functional, i.e. as a linear function on a suitable space of functions. The choices of the appropriate spaces of "test functions" such as $\mathcal{S}$ (there are others which are convenient for different purposes) with their topologies and the whole subject of "generalized functions" or "distributions" was developed by Laurent Schwartz around 1950.

1. What is the k-th derivative of the Dirac delta function?

We may generalize the Heaviside function by considering the function

$$x_+^\lambda := \begin{cases} x^\lambda & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x \le 0 \end{cases} . \tag{2}$$

As a *function* of x, this is defined for all complex values of λ by the rule

$$x^\lambda = e^{\lambda \log x}.$$

For $\lambda = 0$ we get $H(x)$. For $\mathrm{Re}\lambda > -1$ the integral

$$\int_0^\infty f(x) x_+^\lambda dx$$

converges for all $f \in \mathcal{S}$ and so x_+^λ defines a continuous linear function on $\mathcal{S}$. (I am following the standard convention here and not using a complex conjugation; equally well, just consider λ real.) So for these values of λ, we can consider x_+^λ as defined by (2) as an element of $\mathcal{S}'$. We never have any trouble with the convergence of the above integral at infinity. But we have convergence problems at 0 when $\mathrm{Re}\ \lambda \le -1$. We will therefore have to modify the definition (2) for these values of λ.

Notice that for $\mathrm{Re}\ \lambda > 0$ we have

$$\frac{d}{dx}(x_+^\lambda) = \lambda x_+^{\lambda-1} \tag{3}$$

both as functions and as elements of $\mathcal{S}'$.

2. Compute the derivative $\frac{d}{dx}(x_+^\lambda)$ for $-1 < \mathrm{Re}\ \lambda < 0$. In fact, show that

$$\langle (x_+^\lambda)', f \rangle = \int_0^\infty \lambda x^{\lambda-1} [f(x) - f(0)] dx.$$

[Hint: Write $\int_0^\infty x^\lambda f'(x) dx$ as $\lim_{\epsilon \to 0} \int_\epsilon^\infty x^\lambda f'(x) dx$ and do an integration by parts with $f'(x)dx = du$, $u = f(x) + C$, and $v = x^\lambda$. Choose C appropriately so as to be able to evaluate the limit.]

Notice that the function $f(x) - f(0)$ will not belong to the space $\mathcal{S}$ if $f(0) \neq 0$. What makes the above integral converge at infinity is that $x^{\lambda-1}$ vanishes rapidly enough at infinity if $\mathrm{Re}\ \lambda < 0$.

This suggests the following strategy: For Re $\lambda > -1$ we have

$$\int_0^1 x^\lambda dx = \frac{1}{\lambda + 1}.$$

So we can write

$$\langle x_+^\lambda, f\rangle = \int_0^1 x_+^\lambda [f(x) - f(0)]dx + \int_1^\infty x^\lambda f(x)dx + \frac{f(0)}{\lambda+1}. \qquad (4)$$

Notice that the right hand side of this equation makes sense for all λ such that Re $\lambda > -2$ with the exception of the single point $\lambda = -1$. (In the language of complex variable theory we would say that the left hand side is a meromorphic function of λ with a pole at -1 with residue δ.) We will therefore take the right hand side of (4) as a new definition of x_+^λ valid for all $\lambda \neq -1$ such that Re $\lambda > -2$.

3. Show that in the range $-2 <$ Re $\lambda < -1$ we can write the above expression as

$$\int_0^\infty x^\lambda [f(x) - f(0)]dx.$$

Conclude that we can write the result of Problem 2 as asserting the validity of (3) on the range Re $\lambda > -2, \lambda \neq -1$.

4. Obtain a formula for x_+^λ which extends its range into the region

$$\text{Re } \lambda > -n-1, \quad \lambda \neq -1, -2, \cdots, -n$$

for any positive integer n, and obtain a simpler formula valid on the range $-n-1 <$ Re $\lambda < -n$ to conclude the validity of (3) on the strip $-n <$ Re $\lambda < -n-1$.

2 Differentiating under the limit.

Let $\{\ell_n\}$ be a weakly convergent sequence of elements of $\mathcal{S}'$. By definition, this means that there is an $\ell \in \mathcal{S}'$ such that for every $f \in \mathcal{S}$

$$\langle \ell_n, f\rangle \to \langle \ell, f\rangle.$$

But then

$$\ell_n' \to \ell'$$

as well, since by definition

$$\langle \ell_n', f\rangle = -\langle \ell_n, f'\rangle \to -\langle \ell, f'\rangle = \langle \ell', f\rangle.$$

Similarly, if we have a series $\ell_1 + \ell_2 + \cdots$ of generalized functions which converges in this weak sense, we may differentiate term by term to obtain the derivative of the sum.

For example, let h be a piecewise smooth function which is periodic of period 2π. Its Fourier series converges at all points, and hence (by say the dominated convergence theorem) it converges as a series in the weak topology of $\mathcal{S}'$. We know (from Gibbs!) that

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots = \frac{1}{2}(\pi - x) \quad 0 < x < 2\pi$$

and periodic of period 2π on $\mathbf{R}$.

5. Differentiate this formula and conclude that

$$\cdots + e^{-2ix} + e^{-ix} + 1 + e^{ix} + e^{2ix} + \cdots = 2\pi \sum_{-\infty}^{\infty} \delta(x - 2\pi n).$$

and from this conclude the Poisson summation formula.

Let

$$v(x,t) := \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad t > 0$$

6. Show that

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) v \equiv 0$$

and

$$\lim_{t \to 0} v(x,t) = \delta$$

in the sense of generalized functions.

3 Extending the domain of definition.

We may define the **support** of a generalized function $\ell \in \mathcal{S}'$ as follows: We say that ℓ vanishes on an open set U if $\langle \ell, f \rangle = 0$ for all f with $\mathrm{supp}(f) \subset U$. It is not hard to verify that there is a maximal open set with this property, and the support of ℓ is defined to be the complement of this set. Put contrapositively, what this says if that $x \in \mathrm{supp}\ \ell$ if for every neighborhood U of x there is an $f \in \mathcal{S}$ with supp $(f) \subset U$ such that $\langle \ell, f \rangle \neq 0$.

For example, the generalized functions x_+^λ all have their support on the set of non-negative real numbers.

Suppose that we have a function f which does not necessarily belong to $\mathcal{S}$, but which has the property that there is a function ϕ which is identically one in a neighborhood of $\mathrm{supp}(\ell)$ and such that

$$\phi f \in \mathcal{S}.$$

We could then define

$$\langle \ell, f \rangle := \langle \ell, \phi f \rangle$$

and this definition will not depend on ϕ. For example, the function e^{-x} does not belong to $\mathcal{S}$ because of the blow up at $-\infty$. But nevertheless we may apply the x_+^λ to it. In fact Euler's Gamma function defined by

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} e^{-x} dx$$

can be thought of as applying the generalized function $x_+^{\lambda-1}$ to the function e^{-x} which is valid even though $Ee^{-x} \notin \mathcal{S}$. The results of Problem 4 can be thought of as a generalization of the "analytic continuation of the Gamma function".

To be continued.