Problem set 8

Math 212a

November 30, 2000 due Dec. 11

We continue our study of generalized functions.

Contents

1 The spaces $\mathcal{D}$ and $\mathcal{D}'$.

In the previous problem set we defined a generalized function to be a continuous linear function on the Schwartz space $\mathcal{S}$, and denoted the space of generalized functions by $\mathcal{S}'$. For various reasons (especially when we want to extend the theory to manifolds) it is convenient to study other spaces of "test functions" and "generalized functions".

Let $\mathcal{D}$ denote the space of infinitely differentiable functions each of which vanishes outside a compact set K (which may depend on the function). A sequence $f_n \in \mathcal{D}$ is said to converge to an $f \in \mathcal{D}$ if there is a fixed compact set K such that $\operatorname{supp} f_n \subset K$ for all n, and then such that the f_n converge to f uniformly, together with uniform convergence of all the derivative. Then $\mathcal{D}'$ consists of all linear functions which are continuous relative to this notion of convergence.

We should really write $\mathcal{D}(\mathbf{R})$ if we are thinking of functions of one real variable, but we could equally well consider functions of n variables, in which case we would write $\mathcal{D}(\mathbf{R}^n)$ or write $\mathcal{D}(V)$ where V is any finite dimensional vector space.

For example, if Δ denotes the Laplacian on $\mathbf{R}^3$ (or $\mathbf{R}^n$) then one way of writing Green's theorem is

$$\int_G f\Delta\phi dx = \int_G (\Delta f)\phi dx + \int_{\partial G}\left(f\frac{\partial\phi}{\partial n} - \frac{\partial f}{\partial n}\phi\right) dS \qquad (1)$$

where f and ϕ are smooth functions, G is a region with smooth (or piecewise smooth) boundary ∂G, where dS denotes the surface measure on the boundary and $\partial/\partial n$ denotes normal derivative. If we take both f and ϕ to be in $\mathcal{D}$, and take G so large that both $\operatorname{supp} f$ and $\operatorname{supp} \phi$ lie in the interior of G, so that there are no boundary terms,then (1) becomes

$$\langle \Delta f, \phi \rangle = \langle f, \Delta \phi \rangle. \tag{2}$$

For a generalized function f we take this as the definition of Δf, i.e. Δf is defined to be that generalized function given by

$$\langle \Delta f, \phi \rangle := \langle f, \Delta \phi \rangle \quad \forall \phi \in \mathcal{D}.$$

For example, suppose that g is a smooth function, that G is as in (1), and we define f to be equal to g inside G and to be zero outside G, i.e. $f = \mathbf{1}_G g$. Then (1) says that Δf is the sum of the function $(\Delta g)\mathbf{1}_G$ plus two other terms supported on the boundary.

In three dimensions, consider the function $1/r$ (where r is the distance from the origin) which is smooth away from the origin, and (by direct computation) satisfies $\Delta(1/r) = 0$ there. Since the function $1/r$ is integrable in three dimensions at the origin, it defines an element of $\mathcal{S}'$ and of $\mathcal{D}'$ and so it makes sense to compute $\Delta(1/r)$ according to the above definition.

1. Show that

$$\Delta\left(\frac{1}{r}\right) = -4\pi\delta.$$

[Hint: Apply (1) to the region consisting of $\epsilon \leq r \leq R$ where R is chosen so large that $\operatorname{supp} \phi$ is contained in the open ball of radius R so that there are no terms coming from the outer boundary $r = R$. Compute the terms coming from the inner boundary and let $\epsilon \to 0$.]

2 Cauchy principle value.

Back to one dimension temporarily. The function $x \mapsto 1/x$ is not locally integrable in one dimension, but we do have $\log(|x|)' = 1/x$ at all $x \neq 0$ and $\log|x|$ is locally integrable and so defines a generalized function. We may therefore try to define a generalized function by taking the derivative of $\log|x|$ in the sense of generalized functions.

2. Show that $(\log|x|)' = \operatorname{pv}\left(\frac{1}{x}\right)$ where

$$\langle \operatorname{pv}\left(\frac{1}{x}\right), \phi \rangle = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x} dx.$$

The symbol "pv" stands for principal value, a notion that was introduced by Cauchy. Also compute the second derivative of $\log|x|$.

3. Show that if $U \in \mathcal{D}'$ has derivative zero, then U is a constant, i.e. $\langle U, \phi \rangle = C \int_{\mathbf{R}} \phi dx$. [Hint: let $\mathcal{D}_0$ denote the set of all $\psi \in \mathcal{D}$ such that $\int_{\mathbf{R}} \psi dx = 0$. First show that U vanishes on $\mathcal{D}_0$. Then choose an element $\theta \in \mathcal{D}$ with $\int_{\mathbf{R}} \theta dx = 1$. Write every $\phi \in \mathcal{D}$ as $\phi = \psi + a\theta$ where $a = \int_{\mathbf{R}} \phi dx$.]

4. Use the preceding problem to show the following: Suppose that f is a continuous function whose derivative in the sense of generalized functions is a continuous function. I.e. assume that $-\int_{\mathbf{R}} f\phi' dx =: \langle f', \phi \rangle = \int_{\mathbf{R}} g\phi dx =: \langle g, \phi$. Show that f is in fact continuously differentiable as a function and that $f' = g$ in the usual sense.

3 The tensor product of two generalized functions.

Let X and Y be two finite dimensional vector spaces over the real numbers, so that $X \times Y = X \oplus Y$ is again a finite dimensional vector space over the real number. We will write the typical point of $X \times Y$ as (x, y). We can consider the space $\mathcal{D}(X \times Y)$ which consists of all infinitely differentiable functions of compact of compact support on $X \times Y$. We can also consider the space $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ which consists of all finite linear combinations of expressions of the form $\phi\psi$ where $\phi \in \mathcal{D}(X)$ and $\psi \in \mathcal{D}(Y)$. Any such expression defines a function on $X \times Y$ by the rule

$$(\phi\psi)(x, y) = \phi(x)\psi(y),$$

so we have an injection

$$\mathcal{D}(X) \otimes \mathcal{D}(Y) \mapsto \mathcal{D}(X \times Y)$$

which allows us to think of $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ as a subspace of $\mathcal{D}(X \times Y)$. The Stone-Weierstrass theorem, or the original Weierstrass approximation theorem guarantees that $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ is dense in $\mathcal{D}(X \times Y)$.

Now let $f \in \mathcal{D}'(X)$ and $g \in \mathcal{D}'(Y)$ be generalized functions. Let $\phi \in \mathcal{D}(X \times Y)$. For each fixed x, the function $\phi(x, \cdot) : y \mapsto \phi(x, y)$ belongs to $\mathcal{D}(Y)$ and so we can apply g to it to obtain the function $x \mapsto \langle g, \phi(x, \cdot) \rangle$. This is a function of x, and the continuity properties of g imply that this function belongs to $\mathcal{D}(X)$. We then may apply f to the function $x \mapsto \langle g, \phi(x, \cdot) \rangle$ to obtain a number. The notation is getting cumbersome, so we will shorten it and write the final result as

$$\langle f, \langle g, \phi \rangle \rangle.$$

We then define $f \otimes g$ to be this generalized function. In other words we define $f \otimes g \in \mathcal{D}'(X \times Y)$ by

$$\langle f \otimes g, \phi \rangle = \langle f, \langle g, \phi \rangle \rangle. \tag{3}$$

If $\phi = \tau \otimes \eta$ where $\tau \in \mathcal{D}(X)$ and $\eta \in \mathcal{D}(Y)$ then it is clear from the definition that

$$\langle f \otimes g, \tau\eta \rangle = \langle f, \tau \rangle \langle g, \eta \rangle.$$

This shows that on function of the form $\tau\eta$ it would not have made any difference in the definition of $f \otimes g$ had we done things in the reverse order, i.e. first apply f to the function $x \mapsto \phi(x, y)$ and then apply g to the resulting function of y. But since $\mathcal{D}(X) \otimes \mathcal{D}(Y)$ is dense in $\mathcal{D}(X \times Y)$ it follows that doing things in the reverse order yields the same answer on all of $\mathcal{D}(X \times Y)$. This is a sort of "generalized function version" of Fubini's theorem.

Similarly, if we have three vector spaces X, Y and Z and $f \in \mathcal{D}(X)$, $g \in \mathcal{D}(Y)$, $h \in \mathcal{D}(Z)$ we can form $f \otimes (g \otimes h)$ and $(f \otimes g) \otimes h$ and verify that they give the same element of $\mathcal{D}(X \times Y \times Z)$.

It is easy to check directly from the definition that

$$\operatorname{supp}(f \otimes g) = \operatorname{supp}(f) \times \operatorname{supp}(g)$$

as a subset of $X \times Y$.

Suppose that $X = Y = Z$, and to fix the ideas (and hopefully to get the powers of 2π right, although I am not all that optimistic) that they all equal $\mathbf{R}$. If f and g were elements of L_1 we defined their convolution $f \star g$ as

$$(f \star g)(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) g(u - x) ds.$$

If we think of this as a generalized function and apply it to ϕ we get

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x) g(u - x) dx \phi(u) du = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x) g(y) \phi(x + y) dx dy.$$

Therefore we would like to define the convolution $f \star g$ of any two elements of $\mathcal{D}'(\mathbf{R})$ by

$$\langle f \star g, \phi \rangle = \frac{1}{\sqrt{2\pi}} \langle f \otimes g, \phi(x + y) \rangle \tag{4}$$

Here $\phi \in \mathcal{D}(\mathbf{R})$ and $\phi(x + y)$ denotes the function of two variables given by $(x, y) \mapsto \phi(x + y)$. The trouble is that $\phi(x + y)$ does not have compact support as a function of two variables. Indeed it is constant on any "anti-diagonal" line $x + y = a$. So the $\operatorname{supp} \phi(x + y)$ is the anti-diagonal strip consisting of all (x, y) such that $x + y \in \operatorname{supp}(\phi)$. So we can not use (4) in general. But by the remarks of the last section of the preceding exercise set, we can use this definition if we know that $\operatorname{supp}(f) \times \operatorname{supp}(g)$ intersects every anti-diagonal strip (of bounded width) in a compact set.

This will happen, for example, if

- Either f of g has compact support. For then $\operatorname{supp}(f \otimes g)$ will be a horizontal or a vertical strip, which then meets any anti-diagonal strip of bounded width in a compact set. Or

- $\operatorname{supp}(f) \subset [a, \infty)$ and $\operatorname{supp}(g) \subset [b, \infty)$ for then $\operatorname{supp}(f \otimes g)$ is contain in the (infinite) rectangle $[a, \infty) \times [b, \infty)$ which also intersects any bounded anti-diagonal strip in a compact set.

As an illustration of the first case, consider what happens if we take $f = \delta$. Then

$$\langle \delta \star g, \phi \rangle = \frac{1}{\sqrt{2\pi}} \langle \delta \otimes g, \phi(x+y) \rangle = \frac{1}{\sqrt{2\pi}} \langle g, \langle \delta, \phi(x+y) \rangle \rangle = \frac{1}{\sqrt{2\pi}} \langle g, \phi \rangle.$$

In other words,

$$(\sqrt{2\pi}\delta) \star g = g$$

for any $g \in \mathcal{D}'(\mathbf{R})$. Convolution with the element $\sqrt{2\pi}\delta$ is the identity operator.

Similarly, we can define the convolution of two generalized function in n dimensions, the unfortunate factor $\frac{1}{\sqrt{2\pi}}$ (which was determined by our conventions for the Fourier transform) being replaced by the factor $\frac{1}{(2\pi)^{n/2}}$. The second sufficient condition for the definition of the convolution is replaced by the condition that $\text{supp}(f)$ and $\text{supp}(g)$ are both contained in the same type of "orthant", for example in the "first quadrant" (or any translate thereof) in two variables.

Let D be any differential operator with constant coefficients. Define D^* to be the formal adjoint of D obtained by integration by parts and ignoring the boundary terms. So, for example, if D is a homogeneous differential operator of order k, then $D^* = (-1)^k D$. For instance $\Delta^* = \Delta$ for the Laplacian. The operator D is then defined on generalized functions by

$$\langle Df, \phi \rangle := \langle f, D^*\phi \rangle.$$

5. Show that

$$D(f \star g) = (Df) \star g = f \star Dg$$

whenever $f \star g$ is defined. In particular, in three dimensions, if f is any generalized function then

$$u = \left(\frac{c}{r}\right) \star f$$

is a solution to

$$\Delta u = f$$

where (I hope) $c = -\sqrt{\frac{\pi}{2}}$.

4 The wave equation in one dimension.

The convolution, where it is defined, is a continuous function of its variable. For example if f_t is a family of generalized functions (say of compact support) which depend differentiably on a parameter t, then $f_t \star g$ depends differentiably on t and we have

$$\frac{d}{dt}(f_t \star g) = \frac{d}{dt} f_t \star g.$$

For example, consider the function $F = \frac{1}{2}\mathbf{1}_{\mathbf{C}}$ where $\mathbf{C}$ is the "forward cone $|x| \leq t$ in the (x,t) plane. Let $F_t(x) = F(x,t)$. So

$$F_t(x) = \frac{1}{2} \quad \text{if } |x| \leq t$$

and equals zero otherwise. For each fixed t this is a function of x which we can consider as a generalized function of x, and hence apply $\frac{d}{dx}$ to it. By abuse of language we will denote this operation by $\frac{\partial}{\partial x}F$. On the other hand, we can differentiate F_t with respect to t to get a generalized function again depending on t. By abuse of language we will denote this operation by $\frac{\partial}{\partial t}$.

6. Verify that

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{1}{2}\delta(x+t) - \frac{1}{2}\delta(x-t) \\
\frac{\partial^2 F}{\partial x^2} &= \frac{1}{2}\delta'(x+t) - \frac{1}{2}\delta'(x-t) \\
\frac{\partial F}{\partial t} &= \frac{1}{2}\delta(x+t) + \frac{1}{2}\delta(x-t) \\
\frac{\partial^2 F}{\partial t^2} &= \frac{1}{2}\delta'(x+t) - \frac{1}{2}\delta'(x-t).
\end{aligned}$$

From the second and fourth equation we get

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial F}{\partial x^2}$$

which says that F is a solution of the wave equation. From its definition we have $F_t \to 0$ as $t \to 0$ in the sense of generalized functions. From the third equation we get

$$\lim_{t\to 0} \frac{\partial F}{\partial t} = \delta(x).$$

So if we set $E = \sqrt{2\pi}F$ so as to cancel the stupid factor of $1/\sqrt{2\pi}$ which is our convention for convolution, we see that

$$w = E \star v$$

is the solution to the wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}$$

with the initial conditions

$$w(x,0) = 0, \qquad \frac{\partial w}{\partial t}(x,0) = v.$$

This is true for any generalized function v. If v is actually a locally integrable function then this reads

$$w = \int_{-\infty}^{\infty} F(s,t)v(x-s)ds = \frac{1}{2}\int_{-t}^{t} v(x-s)ds$$

or

$$w(x,t) = \frac{1}{2}\int_{x-t}^{x+t} v(r)dr.$$

For the sake of completeness we should record here how D'Alembert used this formula to find the solution of the wave equation with general initial conditions

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

First solve the wave equation with the initial conditions $w(x,0) = 0$ and $\frac{\partial w}{\partial}(x,0) = u_0(x)$ so

$$w(x,t) = \frac{1}{2}\int_{x-t}^{x+t} u_0(r)dr.$$

The derivative with respect to t of any solution of the wave equation is again a solution of the wave equation. So

$$f(x.t) := \frac{\partial w}{\partial t}(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)]$$

is a solution of the wave equation with initial conditions

$$f(x,0) = u_0(x), \quad \frac{\partial f}{\partial t}(x,0) = \frac{1}{2}[u_0'(x) - u_0'(x)] = 0.$$

So subtracting f and applying the preceding result gives D'Alembert's formula

$$u(x,t) = \frac{1}{2}[u_0(x+t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(r)dr.$$