Problem set 4

Math 212a

October 12, 2000, Due October 24

1 The Hausdorff dimension of the Sierpinski gasket.

The purpose of the problems in this problem set is to walk through part of the computation of the Hausdorff dimension of the Sierpinski gasket, and some related notions, all extracted from Edgar *Measure, Topology and Fractal Geometry.*

Let F denote the set of (half)-infinite sequences from a 3 letter alphabet. Say the letters are L, U, and R. For each $0 < r < 1$ we can put a metric on F just as in the two letter case:

$$d_r(x,y) = r^{|\alpha|}, \quad \text{where } x = \alpha x',\ y = \alpha y'$$

and the initial letters of x' and y' are different. As in the two letter case, the sets $[\alpha]$ of all sequences with initial string α form a basis of the open sets. For reasons which will soon become clear, we will take $r = \frac{1}{2}$.

On the other hand $[\alpha] = [\alpha L] \cup [\alpha U] \cup [\alpha R]$ so if we define

$$\ell([\alpha]) = \left(\frac{1}{3}\right)^{|\alpha|}$$

then

$$\ell([\alpha]) = m([\alpha L]) + m([\alpha U]) + m([\alpha R]).$$

1. Use these facts to show that $(F, d_{\frac{1}{2}})$ has Hausdorff dimension $\log 3/\log 2$.

The definition of Sierpinski gasket is that it is the compact subset of the unit square which is the unique fixed point in the set of compact subsets of the unit square under the map

$$K \mapsto T_L(K) \cup T_U(K) \cup T_R(K)$$

where the transformations T_L, T_U, T_R of the plane are given by

$$T_L : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y \end{pmatrix}, \quad T_U : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y + \frac{1}{2} \end{pmatrix}, \quad T_R : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}x + \frac{1}{2} \\ \frac{1}{2}y \end{pmatrix}.$$

Recall how this works in the MATLAB program where J stands for the unit square (colored blue). Then

$$T(J) = \begin{matrix} J & \\ J & J \end{matrix}$$

where each J is now a square of half the dimensions of the original square. If we write the first bit in the binary expansion of the points in this set we have

1	J	
0	J	J
	0	1

Notice that all points whose (only) binary expansion of the x and y coordinates have a 1 in the first position are excluded. At the next stage we have

11	J			
10	J	J		
01	J		J	
00	J	J	J	J
	00	01	10	11

2. Convince yourself that the Sierpinski gasket consists of all points $\begin{pmatrix} x \\ y \end{pmatrix}$ in the unit square where x and y have binary expansions such that a 1 never occurs in the same position in the expansion of x and of y.

Consider the map $h : F \to$ the unit square where the i-th letter in an element of F (which is a sequence of letters from the alphabet L, U, R) goes over into the i-th position in the binary expansion of x and y according to the rule

letter	bit of x	bit of y
L	0	0
U	0	1
R	1	0

so, for example,

$$h(LRLUU\dots) = \begin{pmatrix} .01000\dots \\ .00011\dots \end{pmatrix}.$$

3 Show that

$$\|h(s) - h(t)\| \leq d_{\frac{1}{2}}(s, t)$$

where $\|\cdot\|$ denotes Euclidean distance in the plane, but in contrast to the Cantor set case, we do not have bounded decrease for the map h.

4. Conclude from **3** that the Hausdorff dimension of the Sierpinski gasket is $\leq \log 3/\log 2$.

The proof that the Hausdorff dimension of the Sierpinski gasket is exactly $\log 3/\log 2$ is a lot trickier. For a lot of extra credit you might want to try this.

Let $r_1, \ldots, r_n$ be real numbers satisfying $0 < r_i < 1$. We want to think of the r_i as the "ratio of contraction" of a transformation T_i of a metric space X, i.e.

$$d(T_i x, T_i y) \leq r_i d(x, y) \quad \forall\, x, y \in X.$$

Here we are considering the transformation on compact subsets of X given by

$$T(K) = T_1(K) \cup \cdots \cup T_n(K).$$

The collection $(r_1, \ldots, r_n)$ is called the **ratio list** of T. For example, for the Cantor set the ratio list is $(\frac{1}{3}, \frac{1}{3})$ while for Sierpinski gasket the ratio list is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

5. Show that for any ratio list there is a unique non-negative number s such that

$$r_1^s + \cdots + r_n^s = 1.$$

[Hint: Consider the function $f(s) := \sum_i r_i^s$ so $f(0) = n$ and $f(\infty) = 0$. Show that f is strictly decreasing.]

The number s is called the **similarity dimension** of the fixed set of T.

6. Compute the similarity dimension of the Cantor set and of the Sierpinski gasket.