Problem set 9

Math 212a

December 7, 2000 due Dec. 19

Most of this problem set will be devoted to, or make use of, the Fourier transform extended from $\mathcal{S}$ to its dual space, $\mathcal{S}'$. We will need to do things in n-dimensions, and we will, at least in this problem set, try to adhere to the following conventions: x, y, etc will denote a typical point in $\mathbf{R}^n$ thought of as the space of column vectors with n rows, while ξ, η etc. will be points of the dual space, thought of as consisting of row vectors with n columns. (I probably do not have the temperament to consistently denote this space by $\mathbf{R}_n$.) The pairing between ξ and x will be denoted by $\xi \cdot x$ although undoubtedly I will forget the $\cdot$ on occasion. The space $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ is defined just as in one dimension as functions which are differentiable to all order and which vanish at infinity faster that the inverse of any polynomial, and it is a Frechet space. The Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}_n)$ and is given by $\mathcal{F}(f) = \hat{f}$ where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} f(x) e^{-i\xi\cdot x} dx.$$

Other than the change from $1/\sqrt{2\pi}$ in the one dimensional case to $(2\pi)^{-n/2}$ in the n-dimensional case all theorems remain essentially unchanged. In particular $\mathcal{F}$ is a topological isomorphism, has a unitary extension to L_2 and the Fourier inversion formula holds: $\mathcal{F}^{-1}$is given by $\phi \mapsto \check{\phi}$ where

$$\check{\phi}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}_n} \phi(\xi) e^{i\xi\cdot x} d\xi.$$

Recall that the space $\mathcal{S}'$ denotes the space of continuous linear functions on $\mathcal{S}$, and that we denote the value of $T \in \mathcal{S}'$ on $g \in \mathcal{S}$ by

$$\langle T, g \rangle.$$

The space $\mathcal{S}$ embeds into $\mathcal{S}'$ where the linear function corresponding to f is given by integration against f:

$$\langle f, g \rangle = \int_{\mathbf{R}^n} f(x) g(x) dx.$$

Recall from our notes on the Fourier integral the "multiplication formula" which says that

$$\int_{\mathbf{R}^n} \hat{f}(x) g(x) dx = \int_{\mathbf{R}^n} f(x) \hat{g}(x) dx.$$

This reads

$$\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle.$$

But this makes sense for any element of $\mathcal{S}'$. In other words we **define** the Fourier transform $\hat{T}$ of any $T \in \mathcal{S}'$ by

$$\langle \hat{T}, g\rangle := \langle T, \hat{g}\rangle. \tag{1}$$

For example, if δ denotes the (n-dimensional) delta function:

$$\langle \delta, f\rangle = f(0)$$

then its Fourier transform is given by

$$\langle \hat{\delta}, g\rangle = \langle \delta, \hat{g}\rangle = \frac{1}{(2\pi)^{n/2}} \int g(x)dx,$$

or more succinctly

$$\hat{\delta} = \frac{1}{(2\pi)^{n/2}}$$

where the right hand side denotes the constant function which takes on the given constant value.

1. In one dimension, let δ_b denote the delta function at b, so $\langle \delta_b, g\rangle = g(b)$. What is its Fourier transform, and what is the Fourier transform of its derivative, δ_b'?

We will use the notation

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

for an n-tuplet of non-negative integers and

$$\begin{aligned} |\alpha| &:= \sum_i \alpha_i \\ x^\alpha &:= x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\ D^\alpha &:= \frac{\partial^{|\alpha|}}{\partial_{x_1^{\alpha_1}} \cdots \partial x_n^{\alpha_n}} \\ x^2 &:= x_1^2 + \cdots + x_n^2. \end{aligned}$$

2. Show that the Fourier transform of $D^\alpha T$ is $(i\xi)^\alpha \hat{T}$.

3. Suppose that $T \in \mathcal{S}'(\mathbf{R}^n)$ has compact support, so that it makes sense to evaluate T on any C^∞ function, in particular on the function $x \mapsto e^{-i\xi\cdot x}$. Show that $\hat{T}$ is given by the function

$$\hat{T}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle T, e^{-i\xi\cdot x}\rangle$$

generalizing the standard formula for the Fourier transform of elements of $\mathcal{S}$.

For any function g define $\tilde{g}$ by

$$\tilde{g}(x) := g(-x).$$

4. Show that if $f, g, h \in \mathcal{S}$ then

$$\langle f \star g, h\rangle = \langle f, \tilde{g} \star h\rangle.$$

This suggests that we define $T \star g$ for $T \in \mathcal{S}'$ and $g \in \mathcal{S}$ by

$$\langle T \star g, h\rangle := \langle T, \tilde{g} \star h\rangle. \tag{2}$$

5. If $T \in \mathcal{S}'$ and $f \in \mathcal{S}$ show that the Fourier transform of $T \star f$ is $\hat{T} \cdot \hat{f}$.

The rest of this problem set is devoted to the study of the following circle of ideas. Consider the operator

$$H_0 : \quad L_2(\mathbf{R}^3) \to L_2(\mathbf{R}^3)$$

given by

$$H_0 := -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right).$$

Here the domain of H_0 is taken to be those $\phi \in L_2(\mathbf{R}^3)$ for which the differential operator on the right, taken in the distributional sense, when applied to ϕ gives an element of $L_2(\mathbf{R}^3)$.

The operator H_0 has a fancy name. It is called the "free Hamiltonian of non-relativistic quantum mechanics". Strictly speaking we should add "for particles of mass one in units where Planck's constant is one".

The Fourier transform is a unitary isomorphism of $L_2(\mathbf{R}^3)$ into $L_2(\mathbf{R}_3)$ and carries H_0 into multiplication by ξ^2 whose domain consists of those $\hat{\phi} \in L_2(\mathbf{R}_3)$ such that $\xi^2\hat{\phi}(\xi)$ belongs to $L_2(\mathbf{R}_3)$. The operators

$$V(t) : L_2(\mathbf{R}_3) \to L_2(\mathbf{R}_3), \qquad \hat{\phi}(\xi) \mapsto e^{-it\xi^2}\hat{\phi}$$

form a one parameter group of unitary transformations whose infinitesimal generator in the sense of Stone' theorem is operator consisting of multiplication by ξ^2 with domain as given above. [The minus sign before the i in the exponential is the convention used in quantum mechanics. So we write $\exp -itA$ for the one-parameter group associated to the self-adjioint operator A. I apologize for this (rather irrelevant) notational change, but I want to make the notation in this problem set consistent with what you will see in physics books.]

Thus the operator of multiplication by ξ^2, and hence the operator H_0 is a self-adjoint transformation. The operator of multiplication by ξ^2 is clearly non-negative and so every point on the negative real axis belongs to its resolvent set. Let us write a point on the negative real axis as $-\mu^2$ where $\mu > 0$. Then the resolvent is given by multiplication by $-f$ where

$$f(\xi) = f_\mu(\xi) := \frac{1}{\mu^2 + \xi^2}.$$

We can summarize what we "know" so far as follows:

1. The operator H_0 is self adjoint.

2. The one parameter group of unitary transformations it generates via Stone's theorem is
$$U(t) = \mathcal{F}^{-1} V(t) \mathcal{F}$$
where $V(t)$ is multiplication by $e^{-it\xi^2}$.

3. Any point $-\mu^2, \quad \mu > 0$ lies in the resolvent set of H_0 and
$$R(-\mu^2, H_0) = -\mathcal{F}^{-1} m_f \mathcal{F}$$
where m_f denotes the operation of multiplication by f and f is as given above.

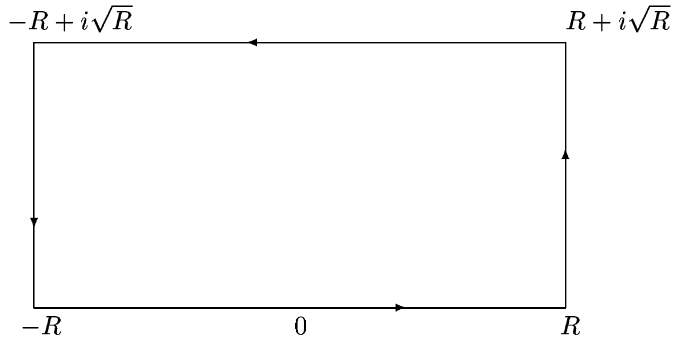
4. If $g \in \mathcal{S}$ and m_g denotes multiplication by g, then the the operator $\mathcal{F}^{-1} m_g \mathcal{F}$ consists of convolution by $\check{g}$. Neither the function $e^{-it\xi^2}$ nor the function f belongs to $\mathcal{S}$, so the operators $U(t)$ and $R(-\mu^2, H_0)$ can only be thought of as convolutions in the sense of generalized functions.

Nevertheless, we will be able to give some slightly more explicit (and very instructive) representations of these operators as convolutions. For example, the next problem will work through the computation of $\check{f}$ and we will find, up to factors of powers of 2π that $\check{f}$ is the function

$$Y_\mu(x) := \frac{e^{-\mu r}}{r}$$

where r denotes the distance from the origin, i.e. $r^2 = x^2$. This function has an integrable singularity at the origin, and vanishes rapidly at infinity. So convolution by Y_μ will be well defined and given by the usual formula on elements of $\mathcal{S}$ and extends to an operator on $L_2(\mathbf{R}^3)$.

The function Y_μ is known as the **Yukawa potential**. Yukawa introduced this function in 1934 to explain the forces that hold the nucleus together. The exponential decay with distance contrasts with that of the ordinary electromagnetic or gravitational potential $1/r$ and, in Yukawa's theory, accounts for the fact that the nuclear forces are short range. In fact, Yukawa introduced a "heavy boson" to account for the nuclear forces. The role of mesons in nuclear physics



was predicted by brilliant theoretical speculation well before any experimental discovery.

6. Since $f \in L_2$ we can compute its inverse Fourier transform as

$$(2\pi)^{-3/2}\check{f} = \lim_{R\to\infty}(2\pi)^{-3}\int_{|\xi|\le R}\frac{e^{i\xi\cdot x}}{\mu^2+\xi^2}d\xi.$$

Here lim means the L_2 limit and $|\xi|$ denotes the length of the vector ξ, i.e. $|\xi| = \sqrt{\xi^2}$ and we will use similar notation $|x| = r$ for the length of x. Let

$$u := \frac{\xi\cdot x}{|\xi||x|}$$

so u is the cosine of the angle between x and ξ. Fix x and introduce spherical coordinates in ξ space with x at the north pole and $s = |\xi|$ so that

$$(2\pi)^{-3}\int_{|\xi|\le R}\frac{e^{i\xi\cdot x}}{\mu^2+\xi^2}d\xi = (2\pi)^{-2}\int_0^R\int_{-1}^1\frac{e^{is|x|u}}{s^2+\mu^2}s^2\,du\,ds = \frac{1}{(2\pi)^2 i|x|}\int_{-R}^R\frac{se^{is|x|}}{(s+i\mu)(s-i\mu)}ds.$$

This last integral is along the bottom of the path in the complex s-plane consisting of the boundary of the rectangle as drawn in the figure.

Show that the remaining three sides of the rectangle make a negligible contribution as $R\to\infty$ and that the limit exists for each fixed x and is given by $e^{-\mu|x|}/(4\pi|x|)$. Conclude that for $\phi\in\mathcal{S}$

$$[(H_0+\mu^2)^{-1}\phi](x) = \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{-\mu|x-y|}}{|x-y|}\phi(y)dy,$$

and since $(H_0+\mu^2)^{-1}$ is a bounded operator on L_2 this formula extends in the L_2 sense to L_2.

The function

$$G_0(x,y;-\mu^2) = \frac{e^{-\mu|x-y|}}{4\pi|x-y|}$$

is called the **free Green's function** in non-relativistic quantum mechanics.

The "explicit" calculation of the operator $U(t)$ is slightly more tricky. The function $\xi \mapsto e^{-it\xi^2}$ is an "imaginary Gaussian", so we expect is inverse Fourier transform to also be an imaginary Gaussian, and then we would have to make sense of convolution by a function which has absolute value one at all points. There are several ways to proceed. One involves integration by parts, and I hope to explain how this works later on in the course in conjunction with the method of stationary phase.

Here I will follow Reed-Simon vol II p.59 and add a little positive term to t and then pass to the limit. In other words, let α be a complex number with positive real part and consider the function

$$\xi \mapsto e^{-\xi^2\alpha}$$

This function belongs to $\mathcal{S}$ and its inverse Fourier transform is given by the function

$$x \mapsto (2\alpha)^{-3/2}e^{-x^2/4\alpha}.$$

(In fact, we verified this when α is real, but the integral defining the inverse Fourier transform converges in the entire half plane Re $\alpha > 0$ uniformly in any Re $\alpha > \epsilon$ and so is holomorphic in the right half plane. So the formula for real positive α implies the formula for α in the half plane.)

We thus have

$$(e^{-H_0\alpha}\phi)(x) = \left(\frac{1}{4\pi\alpha}\right)^{3/2}\int_{\mathbf{R}^3} e^{-|x-y|^2/4\alpha}\phi(y)dy.$$

Here the square root in the coefficient in front of the integral is obtained by continuation from the positive square root on the positive axis. For example, if we take $\alpha = \epsilon + it$ so that $-\alpha = -i(t - i\epsilon)$ we get

$$(U(t)\phi)(x) = \lim_{\epsilon\searrow 0}(U(t-i\epsilon)\phi)(x) = \lim_{\epsilon\searrow 0}(4\pi i(t-i\epsilon))^{-\frac{3}{2}}\int e^{-|x-y|^2/4i(t-i\epsilon)}\phi(y)dy.$$

Here the limit is in the sense of L_2. We thus could write

$$(U(t))(\phi)(x) = (4\pi i)^{-3/2}\int e^{i|x-y|^2/4t}\phi(y)dy$$

if we understand the right hand side to mean the $\epsilon \searrow 0$ limit of the preceding expression.

Actually, as Reed and Simon point out, if $\phi \in L_1$ the above integral exists for any $t \neq 0$, so if $\phi \in L_1 \cap L_2$ we should expect that the above integral is indeed the expression for $U(t)\phi$. Here is their argument: We know that

$$\exp(-i(t-i\epsilon))\phi \to U(t)\phi$$

in the sense of L_2 convergence as $\epsilon \searrow 0$. Recall an old theorem which says that if you have an L_2 convergent sequence you can choose a subsequence which also converges pointwise almost everywhere. So choose a subsequence of ϵ for which this happens. But then the dominated convergence theorem kicks in to guarantee that the integral of the limit is the limit of the integrals.

To sum up: The function

$$P_0(x,y;t) := (4\pi i t)^{-3/2} e^{i|x-y|/4t}$$

is called the **free propagator**. For $\phi \in L_1 \cap L_2$

$$[U(t)\phi](x) = \int_{\mathbf{R}^3} P_0(x,y;t)\phi(y)dy$$

and the integral converges. For general elements ψ of L_2 the operator $U(t)\psi$ is obtained by taking the L_2 limit of the above expression for any sequence of elements of $L_1 \cap L_2$ which approximate ψ in L_2. Alternatively, we could interpret the above integral as the $\epsilon \searrow$ limit of the corresponding expression with t replaced by $t - i\epsilon$.