Solutions: Problem Set 1

1. We have

$$|C(f,n,x)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(x-y)K_n(y)\,dy\right| \le \|f\|_\infty \frac{1}{2\pi}\int_0^{2\pi} |K_n(y)|\,dy$$
$$= \|f\|_\infty \frac{1}{2\pi}\int_0^{2\pi} K_n(y)\,dy = \|f\|_\infty,$$

where the second to last equality follows from the fact that $K_n(y)$ is positive, since

$$K_n(y) = \frac{1}{n+1}\left(\frac{\sin(n+1)y/2}{\sin y/2}\right)^2,$$

and the last equality follows from the fact that the integral of $K_n(y)$ is equal to 2π.

2. We have

$$\frac{L^2-4\pi A}{2\pi^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(2\left(\frac{L}{2\pi}\right)^2 + 4x(t)y'(t)\right)dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(2\left((x'(t))^2+(y'(t))^2\right)+4x(t)y'(t)\right)dt,$$

where the first equality follows from the fact that $A = \int_0^{2\pi} x(t)y'(t)\,dt$ and the second follows from the fact that $(x'(t))^2+(y'(t))^2 = (L/2\pi)^2$. Integration by parts shows that

$$\int_0^{2\pi} x(t)y'(t)\,dt = -\int_0^{2\pi} x'(t)y(t)\,dt,$$

and so

$$\frac{L^2-4\pi A}{2\pi^2} = \frac{1}{2\pi}\int_0^{2\pi}\left(2\left((x'(t))^2+(y'(t))^2\right)+2x(t)y'(t)-2x'(t)y(t)\right)dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(|x'(t)+y(t)|^2+|y'(t)-x(t)|^2-|x(t)|^2-|y(t)|^2\right)dt$$
$$= \sum_{n\in\mathbf{Z}}\left(|ina_n+b_n|^2+|inb_n-a_n|^2+|na_n|^2+|nb_n|^2-|a_n|^2-|b_n|^2\right)$$
$$= \sum_{n\neq 0}\left(|na_n-ib_n|^2+|nb_n+ia_n|^2+(n^2-1)\left(|a_n|^2+|b_n|^2\right)\right),$$

where the second to last equality follows from Parseval's identity. Therefore we conclude that $\frac{L^2-4\pi A}{2\pi^2}\ge 0$, and so $A \le \frac{L^2}{4\pi}$, as desired.

3. The nth Fourier coefficient of e^{ax} is

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx = \frac{1}{2\pi}\frac{1}{a-in}\left(e^{(a-in)\pi}-e^{-(a-in)\pi}\right).$$

Thus the Fourier series of e^{ax} is

$$\frac{1}{2\pi}\sum_{n\in\mathbf{Z}}\frac{1}{a-in}\left(e^{(a-in)\pi}-e^{-(a-in)\pi}\right)e^{inx},$$

which can be rewritten as

$$\frac{e^{a\pi}-e^{-a\pi}}{2\pi a}+\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{a(e^{a\pi}-e^{-a\pi})}{a^2+n^2}\cos(nx)-(-1)^n\frac{n(e^{a\pi}-e^{-a\pi})}{a^2+n^2}\sin(nx).$$

In order to calculate the Fourier series of $\cos(ax)$, we can use the above computations to find the Fourier series for e^{iax} and e^{-iax} and then average them together. This results in

$$\cos(ax) = \frac{1}{\pi a}\sin(a\pi) + \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{2a\sin(a\pi)}{a^2 - n^2}\cos(nx).$$

If we evaluate the Fourier series for $\cos(ax)$ that we just obtained at $x = \pi$ and divide by $\sin(a\pi)/\pi$, we get

$$\pi\cot(a\pi) = \frac{1}{a} + 2a\sum_{n=1}^{\infty}\frac{1}{a^2 - n^2},$$

which is the desired formula.

4. Let a_n be an enumeration of the rationals, and let I_{ij} be an interval of length $1/2^{i+j}$ centered at a_i. Let $O_j = \cup_{i=1}^{\infty} I_{ij}$. Then

$$O_1 \supset O_2 \supset \cdots.$$

Thus, $\cap_{j=1}^{\infty} O_j = \lim_{j\to\infty} O_j = \mathbf{Q}$. Also, each O_j is open and dense, since each O_j is a union of open sets and contains $\mathbf{Q}$, which is dense. So $\mathbf{Q}$ is a countable intersection of open dense sets.

Moreover, $\mathbf{Q}$ has measure zero. This follows since the measure of O_j is less than or equal to

$$\sum_{i=1}^{\infty}|I_{ij}| = \sum_{i=1}^{\infty}\frac{1}{2^{i+j}} = \frac{1}{2^j},$$

and so, since the O_j's satisfy

$$O_1 \supset O_2 \supset \cdots,$$

we see that the measure of their intersection is just the limit as $j \to \infty$ of the measure of O_j, which is bounded by $\lim_{j\to\infty}\frac{1}{2^j}$, and so is just 0.

Consider, for example, the proposition that a real number is rational. This holds quasi-surely, since it holds on the rationals. However, its negation holds almost everywhere, namely on all the irrational numbers, since the rational numbers have measure zero (by the last paragraph), and so anything which is true about their complement is true almost everywhere.

5. We have

$$\begin{aligned} S_n(f,t) = \sum_{j=-n}^{n} a_j e^{ijt} &= \sum_{j=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ijx}\,dx e^{ijt} \\ &= \sum_{j=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{ij(t-x)}\,dx \\ &= \sum_{j=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-x)e^{ijx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-x)\sum_{j=-n}^{n} e^{ijx}\,dx. \end{aligned}$$

Now

$$\sum_{j=-n}^{n} e^{ijx} = e^{-inx}\sum_{j=0}^{2n} e^{ijx} = e^{-inx}\frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} = \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}},$$

as desired.

6. About $t = 0$, we have

$$\frac{1}{\sin t} - \frac{1}{t} = \frac{t - \sin t}{t \sin t}.$$

This last quantity has a removable singularity at 0. We can see, for example, by using L'Hopital's rule and differentiating the numerator and denominator twice that the limit at $t = 0$ is the same as the value of

$$\frac{\sin t}{2\cos t - t \sin t},$$

which is just 0. Thus, in a neighborhood of $t = 0$, $g(t) = \frac{1}{\sin t} - \frac{1}{t}$ is a continuous function if we set $g(0) = 0$. In particular, it is continuous in the region $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. (Note however that g is certainly not continuous at $t = k\pi$ for $k \neq 0$.)

We have

$$\begin{aligned} S_n(f,t) - S_n^*(f,t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-x)(D_n(x) - F_n(x))\, dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-x) g(x/2) \sin\left(\left(n + \frac{1}{2}\right) x\right)\, dx. \end{aligned}$$

By making the change of variables $x \mapsto x + \frac{\pi}{n+\frac{1}{2}}$, we see also that

$$S_n(f,t) - S_n^*(f,t) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(t - x - \frac{\pi}{n+\frac{1}{2}}\right) g\left(\frac{x}{2} + \frac{\pi}{2n+1}\right) \sin\left(\left(n + \frac{1}{2}\right) x\right)\, dx.$$

Taking the average of these two expressions for $S_n(f,t) - S_n^*(f,t)$, we see that

$$S_n(f,t) - S_n^*(f,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(t-x) g(x/2) - f\left(t - x - \frac{\pi}{n+\frac{1}{2}}\right) g\left(\frac{x}{2} + \frac{\pi}{2n+1}\right)\right) \sin\left(\left(n + \frac{1}{2}\right) x\right)\, dx,$$

and so

$$|S_n(f,t) - S_n^*(f,t)| \leq \frac{1}{2} \sup_{x \in [-\pi,\pi]} \left| f(t-x) g(x/2) - f\left(t - x - \frac{\pi}{n+\frac{1}{2}}\right) g\left(\frac{x}{2} + \frac{\pi}{2n+1}\right)\right|.$$

As $n \to \infty$, this last expresion goes to 0 uniformly in t, since both f and g are continuous.