Solutions: Problem Set 2

1. The identity $(x_n - x_m, x_m) = 0$ means that $(x_n, x_m) = \|x_m\|^2$, and so

$$0 \le \|x_n - x_m\|^2 = (x_n - x_m, x_n - x_m) = (x_n - x_m, x_n) = \|x_n\|^2 - \|x_m\|^2.$$

Since this identity holds for all $n > m$, we know that $\|x_m\|^2 \le \|x_n\|^2$ for all $m \le n$.

Since the sequence of values $\{\|x_n\|\}_n$ is bounded and increasing, it must converge to a limit (which is necessarily the greatest lower bound of all the M, as defined in the problem). Thus the sequence $\{\|x_n\|\}_n$ is Cauchy, and so the sequence $\{x_n\}_n$ is Cauchy as well, under the $\|\cdot\|$ norm, since we have the equality

$$\|x_n - x_m\|^2 \le \|x_n\|^2 - \|x_m\|^2.$$

In fact we know that equality holds here, but we only need the inequality to deduce that the sequence $\{x_n\}_n$ must be Cauchy from the fact that the sequence $\{\|x_n\|\}_n$ is Cauchy.

Since $\{x_n\}_n$ is a Cauchy in a complete metric space, it has a limit, x, and from the above observations, $\|x\| = L$, where L is the greatest lower bound of all the M, as defined in the problem.

2. Take the inner product of

$$\frac{1}{n^s}(y_1 + \cdots + y_n)$$

with itself. The cross terms all cancel, since $(y_i, y_j) = 0$ for $i \ne j$. Thus the inner product of this vector with itself is just

$$\left\| \frac{1}{n^s}(y_1 + \cdots + y_n) \right\|^2 = \frac{\|y_1\|^2 + \cdots + \|y_n\|^2}{n^{2s}} \le K n^{1-2s}.$$

This last expression tends to 0 as $n \to \infty$, provided that $s > \frac{1}{2}$.

3. From the definition of a projection we have

$$\pi_A(\mathbf{1}_B) = \frac{(\mathbf{1}_A, \mathbf{1}_B)}{(\mathbf{1}_A, \mathbf{1}_A)} \mathbf{1}_A.$$

We have

$$(\mathbf{1}_A, \mathbf{1}_B) = \int_0^1 \mathbf{1}_A(t)\mathbf{1}_B(t)\, dt = \int_{A\cap B} dt = P(A \cap B),$$

and similarly $(\mathbf{1}_A, \mathbf{1}_A) = P(A)$. Thus

$$\pi_A(\mathbf{1}_B) = \frac{P(A\cap B)}{P(A)} \mathbf{1}_A,$$

as desired.

4. Following the comment after the problem, let

$$p_X(z) = \sum_{n=0}^{\infty} p_n z^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} z^n = e^{-\lambda} e^{\lambda z}.$$

Then $E(X) = p_X'(1) = \lambda$, and $\mathrm{var}(X) = p_X''(1) + p_X'(1) - (p_X'(1))^2 = \lambda$.

To see why $p_X'(1)$ and $p_X''(1) + p_X'(1) - (p_X'(1))^2$ give the mean and variance, respectively, note that

$$p_X'(1) = \sum_{n=0}^{\infty} n p_n = E(X),$$

since X takes on the value n with probability p_n, and

$$\begin{aligned} p''_X(1) + p'_X(1) - (p'_X(1))^2 &= \sum_{n=0}^{\infty} n(n-1)p_n + \sum_{n=0}^{\infty} np_n - (E(X))^2 \\ &= \sum_{n=0}^{\infty} n^2 p_n - (E(X))^2 \\ &= (X, X) - (E(X))^2 = \mathrm{var}(X). \end{aligned}$$

5. The two functions $Y_p(x)$ and $B_p(x)$ can only possibly agree when they are equal to 0 or 1, since $B_p(x)$ never takes on any greater value. The interval where $Y_p(x) = 0$ is $[0, e^{-p}]$, and the interval where $B_p(x) = 0$ is $[0, 1-p]$. Since $1 - p < e^{-p}$, this means that they disagree on the interval $[1-p, e^{-p}]$. To the right of $x = e^{-p}$, they agree once more until $Y_p(x)$ becomes greater than 1, which happens at $x = (1+p)e^{-p}$. Thus they disagree on the union of two intervals,

$$A = [1-p, e^{-p}] \cup [(1+p)e^{-p}, 1].$$

The sum of the lengths of these intervals is thus

$$e^{-p} - (1-p) + 1 - (1+p)e^{-p} = p - pe^{-p} \le p^2,$$

since $1 - p \le e^{-p}$.

6. Since $f \wedge g = 0$, the formulas preceding the problem indicate that $f + g = |f - g|$. Thus, we have

$$f \vee g = \frac{1}{2}(f + g + |f - g|) = f + g,$$

as desired.