Solutions: Problem Set 3

1. We have

$$d_{\text{law}}(X+Z, Y+Z) = \sup_{|f|\le 1} d_f(X+Z, Y+Z) = \frac{1}{2}\sup_{|f|\le 1} |E(f(X+Z)) - E(f(Y+Z))|.$$

Let p_x, q_y, and r_z denote the probabilities that X, Y, and Z assume the values x, y, and z, respectively. Since Z is independent of X, we know that

$$E(f(X+Z)) = \sum_{x,z} f(x+z) p_x r_z,$$

and similarly for $E(f(Y+Z))$.

We can write

$$\begin{aligned} d_{\text{law}}(X+Z, Y+Z) &= \frac{1}{2}\sup_{|f|\le 1}\left|\sum_{x,z} f(x+z)p_x r_z - \sum_{y,z} f(y+z) q_y r_z\right| \\ &= \frac{1}{2}\sup_{|f|\le 1}\left|\sum_z \left(\sum_x f(x+z)p_x - \sum_y f(y+z)q_y\right) r_z\right| \\ &\le \sum_z \frac{1}{2}\sup_{|f|\le 1}\left|\sum_x f(x+z)p_x - \sum_y f(y+z)q_y\right| r_z. \end{aligned}$$

Since the function which takes x to $f(x+z)$ is, for any fixed z, just another function with absolute value less than or equal to 1, the supremum inside this last sum is less than or equal to $d_{\text{law}}(X, Y)$. Thus we see that

$$d_{\text{law}}(X+Z, Y+Z) \le \sum_z d_{\text{law}}(X, Y) r_z = d_{\text{law}}(X, Y),$$

since $\sum_z r_z = 1$.

2. Let $p_X(z) = \sum_{n=0}^{\infty} p_n z^n$ be the generating function for the random variable X which takes on non-negative integer values. Similarly, let $p_Y(z) = \sum_{n=0}^{\infty} q_n z^n$. Notice that

$$p_X(z)p_Y(z) = \sum_{n,m} p_n q_m z^{n+m} = \sum_{k=0}^{\infty}\left(\sum_{n+m=k} p_n q_m\right) z^k.$$

The term inside parentheses in the final sum is just the probability that $X+Y$ will be equal to k, and so we see that the final sum is just $p_{X+Y}(z)$. Thus $p_X(z)p_Y(z) = p_{X+Y}(z)$.

From problem set 2, the generating function for the Poisson distribution with parameter λ is $e^{-\lambda}e^{\lambda x}$. Thus if Z and W are independent Poisson random variables with parameters λ and μ, respectively, the generating function of $Z+W$ is $e^{-(\lambda+\mu)}e^{(\lambda+\mu)x}$. Therefore, $Z+W$ is a Poisson random variable with parameter $\lambda+\mu$.

3. From the statements preceding the problem, we know already that $x \wedge a$ is *an* upper bound for all the $x \wedge x_\alpha$. What remains to show is that it is the least upper bound. So let us assume that y is another upper bound. Then, also from the statements preceeding the problem, we know that if $y \succeq x \wedge x_\alpha$ for all α, then $a \preceq y - x + (x \vee a)$. This last equation is the same as $y \succeq a + x - (x \vee a) = x \wedge a$. Thus, if $y \succeq x \wedge x_\alpha$ for all α, we necessarily have that $y \succeq x \wedge a$, and so $x \wedge a$ is in fact the least upper bound for all the $x_\alpha \wedge x$.

4. We first show that $0 \preceq f_n \preceq \mathbf{1}$. Each e_k is an event, and so $e_k \preceq \mathbf{1}$. Thus $\mathbf{1}$ is an upper bound for all the e_k's, and so f_n, the least upper bound for the first n e_k's, must satisfy $f_n \preceq \mathbf{1}$. Also, since $e_k \succeq 0$ for all k, $f_n \succeq 0$. Thus $0 \preceq f_n \preceq \mathbf{1}$, for all n, as desired.

Next we show that $0 \preceq f_n - f_m \preceq \mathbf{1}$, for $n > m$. We have

$$f_n = \bigvee_{k=1}^{n} e_k = \left(\bigvee_{k=1}^{m} e_k\right) \vee \left(\bigvee_{k=m+1}^{n} e_k\right) = f_m \vee \left(\bigvee_{k=m+1}^{n} e_k\right) \succeq f_m,$$

and so $f_n - f_m \succeq 0$. Also, since $f_m \succeq 0$, we have $f_n - f_m \preceq f_n - 0 \preceq \mathbf{1}$, and so $0 \preceq f_n - f_m \preceq \mathbf{1}$, as desired.

Now we show that the sequence $\{f_n\}_n$ converges. Since $f_n - f_m \preceq \mathbf{1}$, we have

$$0 \leq \|f_n - f_m\|^2 = (f_n - f_m, f_n - f_m) \leq (f_n - f_m, \mathbf{1}) = (f_n, \mathbf{1}) - (f_m, \mathbf{1}). \qquad (*)$$

Since $f_n \succeq f_m$, for $n > m$, and $f_n \preceq \mathbf{1}$, the sequence of real numbers $\{(f_n, \mathbf{1})\}_n$ is increasing bounded above by $(\mathbf{1}, \mathbf{1})$. Thus it is a Cauchy sequence, and so equation (*) implies that $\{f_n\}$ is a Cauchy sequence of elements of the Hilbert space (cf., the reasoning in problem one on the second problem set).

Finally, let e denote the limit of the sequence $\{f_n\}$. Clearly $e \succeq e_k$ for each k (since $f_n \succeq e_k$ for each $n \geq k$, and e is the limit of the f_n's). Also, if x is any other upper bound for all the e_k's, then $x \succeq f_n$ for all n, and so $x \succeq e$ (since e is the limit of the f_n's).

5. Following the hint, let z be the closest element in C to x, where x is the upper bound for the closed space C. Also, let y be an arbitrary element of C. Since y and z are in C, and since C is closed under the $\vee$ operation, we have that $y \vee z \in C$, and so $y \vee z \preceq x$, since x is an upper bound for all of C. From this we see that

$$0 \preceq x - (y \vee z) \preceq x - z,$$

where the first inequality is just a restatement of the forgoing result, and the second comes from the fact that $z \preceq y \vee z$. This equation tells us that $\|x - (z \vee y)\| \leq \|x - z\|$. But, by definition of z we know also that $\|x - z\|^2 \leq \|x - (z \vee y)\|^2$. Thus we have $\|x - z\|^2 = \|x - (z \vee y)\|^2$.

We have

$$\begin{aligned}
\|x - z\|^2 &= \|x - (y \vee z) + (y \vee z) - z\|^2 \\
&= \|x - (y \vee z)\|^2 + 2(x - (y \vee z), (y \vee z) - z) + \|(y \vee z) - z\|^2 \\
&= \|x - z\|^2 + 2(x - (y \vee z), (y \vee z) - z) + \|(y \vee z) - z\|^2.
\end{aligned}$$

Since $x - (y \vee z) \geq 0$, and since $(y \vee z) - z \succeq 0$, we see that $(x - (y \vee z), (y \vee z) - z) \geq 0$. This combined with the last displayed equation tells us that $\|(y \vee z) - z\|^2 = 0$, and so $z = y \vee z$. Because $y \vee z = z$ for any $y \in C$, we see that $z \succeq y$ for any $y \in C$, and so z is the element of C we were trying to find.