Solutions: Problem Set 4

1. Note that if we can show that $(F, d_{\frac{1}{3}})$ has Hausdorff dimension 1, then it will follow that $(F, d_{\frac{1}{2}})$ has Hausdorff dimension $\frac{\log 3}{\log 2}$. This is because

$$(d_{\frac{1}{2}}(x,y))^{\frac{\log 3}{\log 2}} = d_{\frac{1}{3}}(x,y).$$

Thus we need only prove that $(F, d_{\frac{1}{3}})$ has Hausdorff dimension 1.

This proof is entirely similar to the argument in the lecture notes for the computation of the Hausdorff dimension of the Cantor set. Let $\ell([\alpha]) = \left(\frac{1}{3}\right)^{|\alpha|}$. Here $[\alpha]$ is, as in the notes, the set of all sequences that have α as their initial segment. The value $\frac{1}{3}$ is special here in the same way that the value $\frac{1}{2}$ was special for the Cantor set. Indeed, if $k = |\alpha|$, we have

$$\ell([\alpha]) = \left(\frac{1}{3}\right)^k = \left(\frac{1}{3}\right)^{k+1} + \left(\frac{1}{3}\right)^{k+1} + \left(\frac{1}{3}\right)^{k+1} = \ell([\alpha L]) + \ell([\alpha U]) + \ell([\alpha R]).$$

Thus every $[\alpha]$ can be written as the disjoint union of sets in $C_i \in C_\epsilon$ with $\ell([\alpha] = \sum_i \ell(C_i)$. So, as on page 8 in the lecture notes, we see that the type I and type II outer measures constructed from ℓ are equal, and so we can denote them both by m^*. We also see that $m^*([\alpha]) = \ell([\alpha])$. The remainder of the proof follows as on page 11 of the lecture notes (with 3's simply replacing 2's).

2. Let S denote the set of pairs (x, y) such that x and y have binary expansions

$$x = 0.x_1x_2x_3\ldots$$
$$y = 0.y_1y_2y_3\ldots,$$

with the property that x_i and y_i are never both equal to 1 for the same value of i. The maps T_L, T_U, and T_R take (x, y) to the three points

$$(0.0x_1x_2x_3\ldots, 0.0y_1y_2y_3\ldots), \quad (0.0x_1x_2x_3\ldots, 0.1y_1y_2y_3\ldots), \quad \text{and} \quad (0.1x_1x_2x_3\ldots, 0.0y_1y_2y_3\ldots),$$

and so the image of the point (x, y) under the union of these maps is in S.

We need also to see that the union of these maps takes S *onto* S. Notice that the point (x', y') must be in S, where $x' = 0.x_2x_3x_4\ldots$ and $y' = 0.y_2y_3y_4\ldots$. The two digits x_1 and y_1 can not both be 1, and so one of the three maps T_L, T_U, and T_R will take the point (x', y') to the point (x, y). Thus the union of these three maps takes S onto S.

Therefore S is a fixed point of the union of these three maps, and hence the unique fixed point.

3. This problem was misstated. The correct inequality to prove is

$$\|h(s) - h(t)\| \le \sqrt{2}d_{\frac{1}{2}}(s,t).$$

In order to prove this, let s and t be two sequences of letters (with letters from the three letter alphabet $\{L, R, U\}$). We have $d_{\frac{1}{2}}(s,t) = \frac{1}{2^n}$, where n is the number of digits in the initial segment on which s and t agree (possibly $n = 0$). The points $h(s)$ and $h(t)$ are made up of two coordinates whose binary expansions must agree to the first n digits, and so the difference between the first coordinates of these two points can not be greater than $\frac{1}{2^n}$, and similarly for the difference of the second coordinates. Thus the Euclidean distance between these two points in the plane is bounded by

$$\|h(s) - h(t)\| \le \sqrt{\left(\frac{1}{2^n}\right)^2 + \left(\frac{1}{2^n}\right)^2} = \sqrt{2}\frac{1}{2^n} = \sqrt{2}d_{\frac{1}{2}}(s,t).$$

4. For any $A \subseteq F$, we have, from the inequality of problem 3, that

$$(\mathrm{diam}(h(A)))^s \le 2^{\frac{s}{2}}(\mathrm{diam}(A))^s,$$

where the metric used on the left is the Euclidean metric in the plane and that on the right is the $d_{\frac{1}{2}}$ metric on F. From this inequality and the definition of Hausdorff measure, we see that

$$\mathcal{H}_s^*(S) \leq 2^{\frac{s}{2}} \mathcal{H}_s^*(F),$$

where the measure on the left is the outer measure on S and that on the right is the one on F.

From problem 1, we know that the Hausdorff dimension of F is $\frac{\log 3}{\log 2}$, and so from the inequality of the foregoing paragraph, we see that

$$\mathcal{H}_t^*(S) \leq 2^{\frac{s}{2}} \mathcal{H}_t^*(F) < \infty$$

for $t > \frac{\log 3}{\log 2}$. Using Theorem 3 in the lecture notes, we see that this means that the Hausdorff dimension of S can not be greater than $\frac{\log 3}{\log 2}$.

5. As in the hint, let $f(s) = \sum_{i=1}^n r_i^s$. Then $f(0) = n$ and $f(\infty) = 0$, since $0 < r_i < 1$. We also have

$$f'(s) = \sum_{i=1}^n (\log r_i) r_i^s < 0,$$

for $s > 0$, since $0 < r_i < 1$. Thus the continuous function $f(s)$ decreases from $n \geq 1$ to 0 as s goes from 0 to ∞, and so there is a positive value of s for which $f(s) = 1$.

6. For the Cantor set, note that $s = \log 2/\log 3$ satisfies

$$\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1,$$

and for the Sierpinski gasket, note that $s = \log 3/\log 2$ satisfies

$$\left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^s = 1.$$

Thus these are the respective similarlity dimensions of the Cantor set and the Sierpinski gasket.