**Solutions: Problem Set 5**

**1.** We have $g(x) = \left[\frac{1}{x}\right]$, where $[\alpha]$ denotes the greatest integer less than or equal to $\alpha$. Thus the integral of $g$ diverges since

$$\int_0^1 g(x)\,dx = \int_0^1 \left[\frac{1}{x}\right]\,dx = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)k = \sum_{k=1}^{\infty}\frac{1}{k+1} = \infty.$$

**2.** We have

$$\begin{aligned}\int_X\int_Y f(x,y)\,dydx &= \int_0^1\left(\int_0^x \frac{1}{x^2}\,dy - \int_x^1 \frac{1}{y^2}\,dy\right)\,dx\\ &= \int_0^1\left(\frac{1}{x} + 1 - \frac{1}{x}\right)\,dx\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\int_Y\int_X f(x,y)\,dxdy &= \int_0^1\left(\int_0^y -\frac{1}{y^2}\,dx + \int_y^1 \frac{1}{x^2}\,dx\right)\,dy\\ &= \int_0^1\left(-\frac{1}{y} - 1 + \frac{1}{y}\right)\,dx\\ &= -1.\end{aligned}$$

Thus the two integrals are not equal.

We also have

$$\int_{X\times Y} f^+\,dxdy = \int_0^1\int_y^1 \frac{1}{x^2}\,dxdy = \int_0^1\left(\frac{1}{y} - 1\right)\,dy = \infty,$$

and

$$\int_{X\times Y} f^-\,dxdy = \int_0^1\int_x^1 \frac{1}{y^2}\,dydx = \int_0^1\left(\frac{1}{x} - 1\right)\,dy = \infty,$$

and so both of these integrals diverge.

**3.** First we must show htat $\Delta$ is measurable in the product. To see this, notice that

$$\bigcup_{n=1}^{\infty}\left(\bigcup_{i=0}^{n-1}\left[\frac{i}{n},\frac{i+1}{n}\right]\times\left[0,\frac{i}{n}\right)\right) = [0,1]\times[0,1]\setminus\Delta,$$

and so the complement of $\Delta$ is a countable union of sets which are measurable under the product measure. Thus $\Delta$ is measurable under the product measure.

We have

$$\int_X\int_Y f(x,y)\,dydx = \int_X 1\,dx = m(X) = 1,$$

and

$$\int_Y\int_X f(x,y)\,dxdy = \int_Y 0\,dx = 0,$$

and so the integrals disagree.

**4.** The inequality from the hint, namely

$$\operatorname{diam}(A) \geq \max(\|(h,y^+(h)) - (k,y^-(k))\|, \|(h,y^-(h)) - (k,y^+(k))\|),$$

since each element of this maximum is just a distance between points in $A$. We know that

$$\|(h,y^{\pm}(h)) - (k,y^{\mp})\|^2 = \|h-k\|^2 + |y^{\pm}(h) - y^{\mp}(k)|^2,$$

and we know that

$$\max(|y^+(h) - y^-(k)|, |y^-(h) - y^+(k)|) \geq s(h) + s(k),$$

since $2s(h)$ is less than or equal to the length of the interval $[y^-(h), y^+(h)]$, and similarly for $k$. Thus it follows that

$$\begin{aligned}\operatorname{diam}(A) &\geq \max(\|(h, y^+(h)) - (k, y^-(k))\|, \|(h, y^-(h)) - (k, y^+(k))\|) \\ &\geq \max(\|(h, s(h)) - (k, -s(k))\|, \|(h, -s(h)) - (k, s(k))\|).\end{aligned}$$

When the supremum is taken over all $h$'s and $k$'s, the right hand side of this last inequality is seen to be equal to $\operatorname{diam}(St_H(A))$, since $St_H(A)$ is convex, and so we have the desired inequality.

**5.** If $\operatorname{vol}(A) = 0$, then

$$\operatorname{vol}((1-\lambda)A + \lambda B)^{\frac{1}{n}} \geq \operatorname{vol}(\lambda B)^{\frac{1}{n}} = \lambda \operatorname{vol}(B)^{\frac{1}{n}} = (1-\lambda)\operatorname{vol}(A)^{\frac{1}{n}} + \lambda \operatorname{vol}(B)^{\frac{1}{n}}.$$

Thus the inequality holds in this case, and similarly in the case in which $\operatorname{vol}(B) = 0$. We can therefore assume that $A$ and $B$ both have positive volume.

If the inequality holds whenever $A$ and $B$ both have volume 1, then, for more general $A$ and $B$, we have

$$\begin{aligned}\operatorname{vol}((1-\lambda)A + \lambda B)^{\frac{1}{n}} &= \operatorname{vol}\left((1-\lambda)\operatorname{vol}(A)^{\frac{1}{n}} \frac{A}{\operatorname{vol}(A)^{\frac{1}{n}}} + \lambda \operatorname{vol}(B)^{\frac{1}{n}} \frac{B}{\operatorname{vol}(B)^{\frac{1}{n}}}\right) \\ &= c \operatorname{vol}((1-\mu)A' + \mu B'),\end{aligned}$$

where $c = (1-\lambda)\operatorname{vol}(A)^{\frac{1}{n}} + \lambda \operatorname{vol}(B)^{\frac{1}{n}}$, $\mu = \frac{\lambda \operatorname{vol}(B)^{\frac{1}{n}}}{c}$, $A' = A/\operatorname{vol}(A)^{\frac{1}{n}}$, and $B' = B/\operatorname{vol}(B)^{\frac{1}{n}}$. Since $A'$ and $B'$ both have volume 1, our assumption then says that

$$\operatorname{vol}((1-\lambda)A + \lambda B)^{\frac{1}{n}} \geq c = (1-\lambda)\operatorname{vol}(A)^{\frac{1}{n}} + \lambda \operatorname{vol}(B)^{\frac{1}{n}},$$

as desired.

**6.** The desired inequality is the same as

$$((1-\lambda)a^p + \lambda b^p)^{\frac{1}{p}} \geq \left((1-\lambda)a^{-1} + \lambda b^{-1}\right)^{-1},$$

which is the same as

$$(1-\lambda)(a^{-1})^{-p} + \lambda(b^{-1})^{-p} \geq \left((1-\lambda)a^{-1} + \lambda b^{-1}\right)^{-p}.$$

Since the function $x^{-p}$ is concave up for $0 < x < \infty$ and for $p > 0$, this last inequality is true (the chord lies above the graph).

**7.** Any point in $(1-\lambda)k_A(\tau) + \lambda k_B(\tau)$ can be written as $P = (1-\lambda)x + \lambda y$, where $x \in A \cap H(z_A(\tau))$ and $y \in B \cap H(z_B(\tau))$. Since $x \in A$ and $y \in B$, it's clear that $P \in K_\lambda$. Moreover, the $n$th coordinate of $x$ is $z_A(\tau)$ and the $n$th coordinate of $y$ is $z_B(\tau)$. Thus the $n$th coordinate of $P$ is $z_\lambda(\tau)$, and so $P \in H(z_\lambda(\tau))$. Therefore,

$$K_\lambda \cap H(z_\lambda(\tau)) \supset (1-\lambda)k_A(\tau) + \lambda k_B(\tau).$$

As $t$ varies from $z_\lambda(0)$ to $z_\lambda(1)$, the planes $H(t)$ intersect $K_\lambda$, with the endpoints being the points beyond which there is an empty intersection between the planes $H(t)$ and $K_\lambda$. Thus Fubini says that

$$\operatorname{vol}(K_\lambda) = \int_{z_\lambda(0)}^{z_\lambda(1)} \operatorname*{vol}_{n-1}(K_\lambda \cap H(t))\, dt,$$

and this is equal to

$$\int_0^1 \operatorname*{vol}_{n-1}(K_\lambda \cap H(z_\lambda(\tau))) z_\lambda'(\tau)\, d\tau,$$

by the simple change of variables $t = z_\lambda(\tau)$. Using the inclusion of sets from the last paragraph, we see that this says that

$$\begin{aligned}\operatorname{vol}(K_\lambda) &\geq \int_0^1 \operatorname*{vol}_{n-1}\left((1-\lambda)k_A(\tau) + \lambda k_B(\tau)\right) z_\lambda'(\tau)\, d\tau \\ &\geq \int_0^1 \operatorname*{vol}_{n-1}\left((1-\lambda)k_A(\tau) + \lambda k_B(\tau)\right) \left(\frac{1-\lambda}{v_A(z_A(\tau))} + \frac{\lambda}{v_B(z_B(\tau))}\right) d\tau,\end{aligned}$$

where the second inequality follows from the calculations involving $z'(\tau)$ in the discussion preceding the statement of the problem. The full Brunn-Minkowski theorem in $(n-1)$ dimensions says that this means that

$$\begin{aligned}\operatorname{vol}(K_\lambda) &\geq \int_0^1 \operatorname*{vol}_{n-1}\left((1-\lambda)v_A^{\frac{1}{1-n}} + \lambda v_B^{\frac{1}{n-1}}\right)^{n-1}\left(\frac{1-\lambda}{v_A} + \frac{\lambda}{v_B}\right)d\tau \\ &\geq 1,\end{aligned}$$

where the last inequality follows from Problem 6, with $p = 1/(n-1)$, $a = v_A$, and $b = v_B$. This is the desired inequality for regions of volume 1 in $n$ dimensions, and by problem 5, this proves the full Brunn-Minkowski theorem in $n$ dimensions.

**8.** The derivative of $f$ with respect to $\lambda$ is

$$\frac{df}{d\lambda} = \frac{1}{n}\operatorname{vol}(K_\lambda)^{\frac{1}{n}-1}\frac{d\operatorname{vol}(K_\lambda)}{d\lambda} + \operatorname{vol}(A)^{\frac{1}{n}} - \operatorname{vol}(B)^{\frac{1}{n}},$$

From the discussion preceeding the problem, it is seen that

$$\operatorname{vol}(K_\lambda) = (1-\lambda)^n V_0 + n(1-\lambda)^{n-1}\lambda V_1 + \mathrm{O}(\lambda^2),$$

where the big $O$ term is actually a polynomial. Thus,

$$\frac{d\operatorname{vol}(K_\lambda)}{d\lambda} = -n(1-\lambda)^{n-1}V_0 + n(1-\lambda)^{n-1}V_1 + \mathrm{O}(\lambda).$$

Evaluating the derivative of $f$ at $\lambda = 0$, we thus see that it is equal to

$$\left.\frac{df}{d\lambda}\right|_{\lambda=0} = \frac{1}{n}\operatorname{vol}(A)^{\frac{1}{n}-1}(-nV_0 + nV_1) + \operatorname{vol}(A)^{\frac{1}{n}} - \operatorname{vol}(B)^{\frac{1}{n}} = \operatorname{vol}(A)^{\frac{1}{n}-1}V_1 - \operatorname{vol}(B)^{\frac{1}{n}},$$

where we have used the fact that $V_0 = \operatorname{vol}(A)$. This quantity must be greater than or equal to 0, by the Brunn-Minkowski theorem, and so

$$V_1^n \geq \operatorname{vol}\operatorname{vol}(A)^{n-1}\operatorname{vol}(B),$$

as desired.