Solutions: Problem Set 6

1. We have

$$\begin{aligned} M_N(t) &= \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}e^{t^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx \\ &= e^{t^2/2}. \end{aligned}$$

Therefore,

$$E(N) = M'(t)|_{t=0} = te^{t^2/2}|_{t=0} = 0,$$
$$E(N^2) = M''(t)|_{t=0} = (1+t^2)e^{t^2/2}|_{t=0} = 1,$$
$$E(N^3) = M'''(t)|_{t=0} = (3t+t^3)e^{t^2/2}|_{t=0} = 0,$$
$$E(N^4) = M^{(4)}(t)|_{t=0} = (3+6t^2+t^4)e^{t^2/2}|_{t=0} = 3,$$

and $\mathrm{var}(N) = E(N^2) = 1$.

2. We have

$$\begin{aligned} E_{|Z|>\varepsilon s}(Z^2) &= \int_{|z|\geq \varepsilon s} z^2\frac{1}{\sigma\sqrt{2\pi}}e^{-z^2/2\sigma^2}\,dz \\ &= \int_{|z|\geq \frac{\varepsilon s}{\sigma}} (\sigma z)^2\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\frac{\sigma^3}{\varepsilon s}\cdot\frac{\varepsilon s}{\sigma}\int_{|z|\geq \frac{\varepsilon s}{\sigma}} z^2 e^{-z^2/2}\,dz, \end{aligned}$$

where the second line follows by making the change of variables $z \mapsto \sigma z$. This is the same as

$$E_{|Z|>\varepsilon s}(Z^2) = \frac{2}{\sqrt{2\pi}}\frac{\sigma^3}{\varepsilon s}\cdot a\int_{|a|}^{\infty} z^2 e^{-z^2/2}\,dz,$$

where $a = \frac{\varepsilon s}{\sigma}$, and so we only need show that $a\int_{|a|}^{\infty} z^2e^{-z^2/2}\,dz$ is bounded. We can use the gross bound that $z^2e^{-z^2/2} \leq Ce^{-z}$, for some absolute constant C to get that

$$a\int_{|a|}^{\infty} z^2e^{-z^2/2}\,dz \leq a\int_{|a|}^{\infty} Ce^{-z}\,dz = Cae^{-|a|} \leq L,$$

for some absolute constant L. Thus we obtain the desired inequality.

3. For any $\delta > 0$, we have

$$\begin{aligned} \frac{\sigma_k^2}{s_n^2} &= E\left(\frac{X_k^2}{s_n^2}\right) \\ &= E_{|X_k|\leq \delta s_n}\left(\frac{X_k^2}{s_n^2}\right) + E_{|X_k|>\delta s_n}\left(\frac{X_k^2}{s_n^2}\right) \\ &= \frac{1}{s_n^2}E_{|X_k|\leq \delta s_n}(X_k^2) + E_{|X_k|>\delta s_n}\left(\frac{X_k^2}{s_n^2}\right). \end{aligned}$$

Since $E_{|X_k|\leq\delta s_n}(X_k^2)$ is the expected value of X_k^2 when $|X_k| \leq \delta s_n$, it is necessarily less than or equal to $s_n^2\delta^2$. Thus

$$\begin{aligned} \max_{1\leq k\leq n}\frac{\sigma_k^2}{s_n^2} &\leq \delta^2 + \max_{1\leq k\leq n} E_{|X_k|>\delta s_n}\left(\frac{X_k^2}{s_n^2}\right) \\ &\leq \delta^2 + \frac{1}{s_n^2}\sum_{k=1}^{n} E_{|X_k|>\delta s_n}(X_k^2). \end{aligned}$$

Since this final sum goes to 0 as $n \to \infty$, we see that

$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{\sigma_k}{s_n} \le \delta. \tag{$*$}$$

since this holds for any $\delta > 0$, we have

$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{\sigma_k}{s_n} \to 0.$$

Also, using $(*)$ with $\delta < \epsilon^2$, we see that the last term in equation 11 is bounded by a constant multiple of ϵ.
4. If all the X_k's are identically distributed, we can write $X_k = X$ for all k, and see that the left hand side of equation 12 is equal to

$$\lim_{n\to\infty} \frac{1}{s_n^2} \left(nE_{|X|>\varepsilon s_n}(X^2)\right) = \lim_{n\to\infty} \frac{1}{n\sigma^2} \left(nE_{|X|>\varepsilon s_n}(X^2)\right) == \lim_{n\to\infty} \frac{1}{\sigma^2} E_{|X|>\varepsilon\sqrt{n}\sqrt{\sigma}}(X^2).$$

For this problem, we assume that X has a finite variance $\sigma^2 > 0$, and so we know that $E_X(X^2) = \sigma^2 < \infty$. (Note that $E_X(X^2) = \sigma^2$ here becaus $E_X(X) = 0$.) Thus $E_{|X|>\alpha}(X^2)$ must go to 0 as $\alpha \to \infty$, and so as $n \to \infty$, $E_{|X|>\varepsilon\sqrt{n}\sqrt{\sigma}}(X^2)$ must go to 0. This shows that equation 12 is satisfied.