Solutions: Problem Set 7

1. Integrating by parts k times, we see that

$$\int \frac{d^k g}{dx^k} f = (-1)^k \int g \frac{d^k f}{dx^k},$$

and so the kth derivative of δ is given by

$$\delta^{(k)} \colon f \mapsto (-1)^k f^{(k)}(0).$$

2. The derivative of x_+^λ is given by the integrals

$$\langle (x_+^\lambda)', f\rangle = -\int_0^\infty x^\lambda f'(x)\,dx = -\int_0^\infty x^\lambda (f(x)-f(0))'\,dx = \int_0^\infty \lambda x^{\lambda-1}(f(x)-f(0))\,dx,$$

where the integration by parts is permissible because the boundary terms vanish and the resulting integral converges for $-1 < \Re(\lambda) < 0$.

3. When $\Re(\lambda) < -1$, we have $\int_1^\infty x^\lambda f(0)\,dx = \frac{-f(0)}{\lambda+1}$. Thus it is clear that equation (4) from the problem set says that

$$\langle x_+^\lambda, f\rangle = \int_0^\infty x^\lambda (f(x)-f(0))\,dx,$$

in the region $-2 < \Re(\lambda) < -1$.

This formula is the same as

$$\langle \lambda x_+^{\lambda-1}, f\rangle = \int_0^\infty \lambda x^{\lambda-1}(f(x)-f(0))\,dx,$$

in the range $-1 < \Re(\lambda) < 0$, and so this and problem 2 show that $(x_+^\lambda)' = \lambda x_+^{\lambda-1}$ in this range.

4. For any $n \geq 0$, and $\Re(\lambda) > -1$, we can write

$$\begin{aligned}\int_0^\infty x^\lambda f(x)\,dx &= \int_0^1 x^\lambda f(x)\,dx + \int_1^\infty x^\lambda f(x)\,dx \\ &= \int_0^1 x^\lambda \left(f(x) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) dx + \sum_{k=0}^n \frac{f^{(k)}(0)}{k!(\lambda+k+1)} + \int_1^\infty x^\lambda f(x)\,dx.\end{aligned}$$

This last expression is well-defined whenever $\Re(\lambda) > -n-2$ and λ is not a negative integer. Moreover, if $n' > n$, the corresponding formula with n' agrees with the formula for n in the entire range $\Re(\lambda) > -n-2$. Thus we can *define*

$$\langle x_+^\lambda, f\rangle = \int_0^1 x^\lambda \left(f(x) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) dx + \sum_{k=0}^n \frac{f^{(k)}(0)}{k!(\lambda+k+1)} + \int_1^\infty x^\lambda f(x)\,dx,$$

in the region $\Re(\lambda) > -n-2$. Given this definition, we can now note that for $-n-2 < \Re(\lambda) < -n-1$, we have

$$\begin{aligned}\langle x_+^\lambda, f\rangle &= \int_0^1 x^\lambda \left(f(x) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) dx + \int_1^\infty x^\lambda \left(f(x) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) dx \\ &= \int_0^\infty x^\lambda \left(f(x) - \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \right) dx.\end{aligned}$$

From the results of the last paragraph, we see that

$$\begin{aligned}
\langle (x_+^\lambda)', f\rangle = -\langle x_+^\lambda, f'\rangle &= \int_0^\infty x^\lambda \left(f'(x) - \sum_{k=0}^{n} \frac{f^{(k+1)}(0)}{k!} x^k \right) dx \\
&= \int_0^\infty x^\lambda \left(f(x) - \sum_{k=0}^{n+1} \frac{f^{(k)}(0)}{k!} x^k \right)' dx \\
&= \int_0^\infty \lambda x^{\lambda-1} \left(f(x) - \sum_{k=0}^{n+1} \frac{f^{(k)}(0)}{k!} x^k \right) dx \\
&= \langle \lambda x_+^{\lambda-1}, f\rangle,
\end{aligned}$$

in the region $-n-2 < \Re(\lambda) < -n-1$. Thus $(x_+^\lambda)' = \lambda x_+^{\lambda-1}$ in each band $-n-2 < \Re(\lambda) < -n-1$, for $n \geq 0$.

5. The problem requires us to differentiate the function $g(x)$ which is equal to $\frac{1}{2}(\pi - x)$ on $[0, 2\pi]$ and periodic of period 2π. This will be accomplished by computing $-\int_{\mathbf{R}} g(x) f'(x)\, dx$ for Schwartz functions f. We have

$$\begin{aligned}
-\int_{\mathbf{R}} g(x) f'(x)\, dx &= \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \frac{1}{2}(x-\pi) f'(x + 2\pi k)\, dx \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{\pi}{2} f(2\pi(k+1)) + \frac{\pi}{2} f(2\pi k) - \int_0^{2\pi} \frac{1}{2} f(x + 2\pi k)\, dx \right) \\
&= \pi \sum_{k=-\infty}^{\infty} f(2\pi k) - \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, dx.
\end{aligned}$$

Thus we see that the derivative, in the generalized sense, of $g(x)$ is

$$g'(x) = -\frac{1}{2} + \pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k).$$

Since we also have

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin kx,$$

we see from term by term differentiation that, as distributions,

$$g'(x) = \sum_{k=1}^{\infty} \cos kx = -\frac{1}{2} + \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{ikx}.$$

Equating these two expressions for $g'(x)$, we see that, as distributions,

$$\sum_{k=-\infty}^{\infty} e^{ikx} = 2\pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k).$$

We can now note that if we apply the map defined by each side of this last equation to a Schwartz function f, we obtain

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) = 2\pi \sum_{k=-\infty}^{\infty} f(2\pi k),$$

which is the Poisson summation formula.

6. We have

$$\frac{\partial v}{\partial t} = -\frac{1}{4\sqrt{\pi}} t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} + \frac{x^@}{8\sqrt{\pi}} t^{-\frac{5}{2}} e^{-\frac{x^2}{4t}},$$

and

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left(-\frac{x}{4\sqrt{\pi}} t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}}\right) \\ &= -\frac{1}{4\sqrt{\pi}} t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} + \frac{x^2}{8\sqrt{\pi}} t^{-\frac{5}{2}} e^{-\frac{x^2}{4t}}. \end{aligned}$$

Thus it follows that

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) v \equiv 0.$$

For any Schwartz function $f(x)$, we have

$$\begin{aligned} \lim_{t\to 0} \int_{\mathbf{R}} f(x) v(x,t)\, dx &= \lim_{t\to 0} \frac{1}{2\sqrt{\pi t}} \int_{\mathbf{R}} f(x) e^{-\frac{x^2}{4t}}\, dx \\ &= \lim_{t\to 0} \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x\sqrt{t}) e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(0) e^{-\frac{x^2}{2}}\, dx, \end{aligned}$$

where the last line follows by Lebesgue's dominated convergence theorem. Since $\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-\frac{x^2}{2}}\, dx = 1$, this is equal to $f(0)$, and so

$$\lim_{t\to 0} v(x,t) = \delta,$$

as desired.