Solutions: Problem Set 8

1. For $\phi \in \mathcal{D}$, $\langle \Delta\left(\frac{1}{r}\right), \phi\rangle = \langle \frac{1}{r}, \Delta\phi\rangle$, by definition, and

$$\langle \frac{1}{r}, \Delta\phi\rangle = \int_{\mathbf{R}^3} \frac{1}{r}\Delta\phi\, dx = \lim_{\epsilon\to 0^+} \int_{B_{\frac{1}{\epsilon}}\setminus B_\epsilon} \frac{1}{r}\Delta\phi\, dx,$$

where B_a is the ball of radius a centered at the origin. Since ϕ has compact support, we can take ϵ so small that ϕ vanishes on the boundary of $B_{\frac{1}{\epsilon}}$, and so Green's Theorem says that

$$\langle \Delta\left(\frac{1}{r}\right), \phi\rangle = \langle \frac{1}{r}, \Delta\phi\rangle = \lim_{\epsilon\to 0^+} \int_{B_{\frac{1}{\epsilon}}\setminus B_\epsilon} \Delta\left(\frac{1}{r}\right)\phi\, dx - \int_{S_\epsilon} \left(\frac{1}{r}\frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\phi\right) d\sigma,$$

where σ is surface measure on the sphere of radius ϵ, S_ϵ and n is the outward normal to S_ϵ.

For $r \neq 0$,

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = -\frac{1}{r^3} + 3\frac{x^2}{r^5}.$$

Using the similar results for $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}$ and adding, we see that

$$\Delta\left(\frac{1}{r}\right) = 0,$$

for $r \neq 0$. Also, on S_ϵ we have

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2}.$$

Thus,

$$\begin{aligned}\langle \Delta\left(\frac{1}{r}\right), \phi\rangle &= -\lim_{\epsilon\to 0^+} \int_{S_\epsilon} \left(\frac{1}{r}\frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\phi\right) d\sigma \\ &= \lim_{\epsilon\to 0^+} -\int_{S_\epsilon} \frac{1}{r}\frac{\partial\phi}{\partial n} - \lim_{\epsilon\to 0^+} \frac{1}{\epsilon^2}\int_{S_\epsilon} \phi\, d\sigma.\end{aligned}$$

Since ϕ is infinitely differentiable at 0, $\phi(x)$ tends toward $\phi(0)$ as $\epsilon \to 0$ for $x \in S_\epsilon$, and $\frac{\partial\phi}{\partial n}$ remains bounded as $\epsilon \to 0$. Since the surface area of S_ϵ is $4\pi\epsilon^2$, this says that

$$\langle \Delta\left(\frac{1}{r}\right), \phi\rangle == \lim_{\epsilon\to 0} \left(-4\pi\phi(0) + \mathrm{O}(\epsilon)\right) = -4\pi\phi(0),$$

which means that $\Delta\left(\frac{1}{r}\right) = -4\pi\delta$.

2. Using the generalized definition of derivatives, we see that

$$\langle (\log|x|)', \phi\rangle = -\langle \log|x|, \phi'\rangle = -\int_{\mathbf{R}} \log|x|\phi'(x)\, dx = -\int_{\mathbf{R}} \log|x|(\phi(x) - \phi(0))'\, dx.$$

Thus,

$$\begin{aligned}\langle (\log|x|)', \phi\rangle &= -\int_0^\infty \log|x|(\phi(x) - \phi(0))'\, dx + \int_0^\infty \log|x|(\phi(-x) - \phi(0))'\, dx \\ &= \int_0^\infty \frac{\phi(x) - \phi(0)}{x}\, dx - \int_0^\infty \frac{\phi(-x) - \phi(0)}{x}\, dx \\ &= \int_0^\infty \frac{\phi(x) - \phi(-x)}{x}\, dx \\ &= \langle \mathrm{pv}\left(\frac{1}{x}\right), \phi\rangle.\end{aligned}$$

For the second derivative of $\log|x|$, notice that

$$\begin{aligned}\langle (\log|x|)'', \phi\rangle = -\langle (\log|x|)', \phi'\rangle &= -\int_0^\infty \frac{\phi'(x) - \phi'(-x)}{x}\,dx \\ &= -\int_0^\infty \frac{(\phi(x) + \phi(x))'}{x}\,dx \\ &= -\int_0^\infty \frac{(\phi(x) + \phi(x) - 2\phi(0))'}{x}\,dx \\ &= \int_0^\infty \frac{\phi(x) + \phi(x) - 2\phi(0)}{x^2}\,dx.\end{aligned}$$

3. If $\psi \in \mathcal{D}_0$, we have

$$\Psi(x) = \int_{-\infty}^x \psi(t)\,dt \in \mathcal{D}.$$

Indeed, $\Psi(x)$ has compact support, since $\psi \in \mathcal{D}_0$, and it is infinitely differentiable, since ψ is. Let $\theta \in \mathcal{D}$ be a fixed function with the property that $\int_{\mathbf{R}} \theta\,dx = 1$. For any $\phi \in \mathcal{D}$, we have

$$\phi = \psi + a\theta,$$

where $\psi = \phi - a\theta$ and $a = \int_{\mathbf{R}} \phi\,dx$. Notice that such a ψ lies in $\mathcal{D}_0$. Thus it has an antiderivative $\Psi \in \mathcal{D}$, as constructed above, and so

$$\phi = \Psi' + a\theta.$$

Now, since U has derivative 0,

$$\langle U, \phi\rangle = \langle U, \Psi'\rangle + a\langle U, \theta\rangle = -\langle U', \Psi\rangle + a\langle U, \theta\rangle = a\langle U, \theta\rangle = \langle C, \phi\rangle,$$

where $C = \langle U, \theta\rangle$ is a constant since θ is fixed, and where the fact that $a = \int_{\mathbf{R}} \phi\,dx = \langle \phi, 1\rangle$ has been used.

4. Since g is a continuous function, it has a continuous antiderivative, say $G(x) = \int_0^x g(t)\,dt$. We know that g is the distributional derivative of f, and so

$$\langle f', \phi\rangle = \langle g, \phi\rangle = \langle G', \phi\rangle,$$

which means that

$$\langle (f - G)', \phi\rangle = 0,$$

for all ϕ. Thus $U = f - G$ is of the type addressed by problem 3, i.e., its derivative vanishes, and so the result of problem 3 says that $f - G$ is equal to a constant, in the sense of distributions. Since both f and G are continuous, this means that in fact $f - G$ is equal to a constant in the usual sense, and so $f = G + C$ for some constant C (again, in the pointwise sense). By construction, G is continuously differentiable, and so f is therefore continuously differentiable.

5. We have

$$\begin{aligned}\langle D(f \star g), \phi\rangle = \langle f \star g, D^*\phi\rangle &= \frac{1}{\sqrt{2\pi}}\langle f \otimes g, D^*\phi(x+y)\rangle = \frac{1}{\sqrt{2\pi}}\langle f, \langle g, D^*\phi(x+y)\rangle\rangle \\ &= \frac{1}{\sqrt{2\pi}}\langle f, \langle Dg, \phi(x+y)\rangle\rangle = \frac{1}{\sqrt{2\pi}}\langle f \otimes Dg, \phi(x+y)\rangle\rangle \\ &= \langle f * Dg, \phi\rangle\rangle.\end{aligned}$$

Thus $D(f * g) = f * Dg$, and similarly $D(f * g) = Df * g$.

If $u = \left(\frac{c}{r}\right) * f$, then

$$\Delta u = \Delta\left(\frac{c}{r}\right) * f = -4\pi c\delta * f,$$

where we have used the result from problem 1. Since $\delta * f = (2\pi)^{-\frac{3}{2}} f$ in three dimensions, given our conventions for the 2π's, we see that by taking $c = -\sqrt{\frac{\pi}{2}}$,

$$\Delta u = f.$$

6. The problem should have defined $F_t(x) = \frac{1}{2}$ for $|x| \leq |t|$, not just $|x| \leq t$. In this case,

$$F_t(x) = \frac{1}{2} \left(H(x+t) - H(x-t) \right),$$

where $H(u)$ is the usual Heaviside function, which is 1 when u is positive and zero otherwise. Thus

$$\begin{aligned}
\frac{\partial F_t(x)}{\partial x} &= \frac{1}{2} \left(\delta(x+t) - \delta(x-t) \right), \\
\frac{\partial^2 F_t(x)}{\partial x^2} &= \frac{1}{2} \left(\delta'(x+t) - \delta'(x-t) \right), \\
\frac{\partial F_t(x)}{\partial t} &= \frac{1}{2} \left(\delta(x+t) + \delta(x-t) \right),
\end{aligned}$$

and

$$\frac{\partial^2 F_t(x)}{\partial x^2} = \frac{1}{2} \left(\delta'(x+t) - \delta'(x-t) \right),$$

as desired.