Solutions: Problem Set 9

1. We have

$$\langle \widehat{\delta_b}, \varphi \rangle = \langle \delta_b, \hat{\varphi} \rangle = \hat{\varphi}(b) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ixb} \varphi(x)\, dx = \frac{1}{\sqrt{2\pi}} \langle e^{-ixb}, \varphi \rangle.$$

Thus the Fourier transform of δ_b is $\frac{1}{\sqrt{2\pi}} e^{-ixb}$. For its derivative, we have

$$\langle \widehat{\delta_b'}, \varphi \rangle = \langle \delta_b', \hat{\varphi} \rangle = -\langle \delta_b, (\hat{\varphi})' \rangle.$$

Now

$$(\hat{\varphi})' = \frac{d}{d\xi} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \varphi(x)\, dx \right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi}(-ix)\varphi(x)\, dx,$$

and so this function evaluated at b is

$$\langle \delta_b, (\hat{\varphi})' \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ixb}(-ix)\varphi(x)\, dt.$$

This means that

$$\langle \widehat{\delta_b'}, \varphi \rangle = \frac{1}{\sqrt{2\pi}} \langle e^{-ibx}(ix), \varphi \rangle,$$

and so the Fourier transform of δ_b' is $\frac{ix}{\sqrt{2\pi}} e^{-ibx}$.

2. We have

$$\langle \widehat{D^\alpha T}, \varphi \rangle = (-1)^\alpha \langle T, D^\alpha \hat{\varphi} \rangle,$$

where we have used multiple integrations by parts. If φ is Schwartz, so will $\hat{\varphi}$ be, and so we can use Lebesgue's dominated convergence theorem to pass the derivatives under the integral sign in the definition of $\hat{\varphi}$. This yields

$$\begin{aligned} D^\alpha \hat{\varphi} &= \frac{1}{(2\pi)^{n/2}} D^\alpha \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} D^\alpha(e^{-ix\cdot\xi}) \varphi(x)\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-ix\cdot\xi}(-ix)^\alpha \varphi(x)\, dx. \end{aligned}$$

Thus $(-1)^\alpha D^\alpha \hat{\varphi} = \hat{\psi}$, where $\psi(x) = (ix)^\alpha \varphi(x)$, and so

$$\langle \widehat{D^\alpha T}, \varphi \rangle = \langle T, \hat{\psi} \rangle = \langle \hat{T}, (ix)^\alpha \varphi(x) \rangle = \langle (ix)^\alpha \hat{T}, \varphi \rangle,$$

as desired.

3. We have

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle = \frac{1}{(2\pi)^{n/2}} \left\langle T, \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx \right\rangle = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} \langle T, e^{-x\cdot\xi} \rangle \varphi(x)\, dx,$$

which means that

$$\hat{T}(x) = \frac{1}{(2\pi)^{n/2}} \langle T, e^{-ix\cdot\xi} \rangle.$$

4. We have

$$\begin{aligned} \langle f * g, h \rangle &= \int \int f(y)g(x-y)\, dy\, h(x)\, dx \\ &= \int \int g(x-y)h(x)\, dx\, f(y)\, dy \\ &= \int \int \tilde{g}(y-x)h(x)\, dx\, f(y)\, dy \\ &= \langle f, \tilde{g} * h \rangle. \end{aligned}$$

5. The Fourier transform of $T * f$ is given by

$$\langle \widehat{T * f}, \varphi \rangle = \langle T * f, \hat{\varphi} \rangle = \langle T, \tilde{f} * \hat{\varphi} \rangle.$$

We can write $f * \hat{\varphi}$ as the Fourier transform of its inverse Fourier transform, and its inverse Fourier transform is given by first applying the ~ operator and then applying the Fourier tranform. Thus the inverse Fourier tranform of $\tilde{f} * \hat{\varphi}$ is

$$((\tilde{f} * \hat{\varphi})\check{\ })\hat{\ } = \hat{f} \cdot \varphi.$$

We see therefore that

$$\langle \widehat{T * f}, \varphi \rangle = \langle \widehat{T * f}, \varphi \rangle = \langle T, \widehat{\hat{f} \cdot \varphi} \rangle = \langle \hat{T}, \hat{f} \cdot \varphi \rangle.$$

This is the same thing as

$$\langle \widehat{T * f}, \varphi \rangle = \langle \hat{T} \cdot \hat{f}, \varphi \rangle.$$

6. Let $f(s) = \frac{se^{is|x|}}{(s+i\mu)(s-i\mu)}$. For $\Re(s) = \pm R$, $f(s) = \mathrm{O}(1/R)$, and for $\Im(s) = i\sqrt{R}$, $f(s) = \mathrm{O}(e^{-|x|\sqrt{R}})$. These are the bounds that apply along the vertical and uper horizontal sides, respectively, of the rectangle shown for the problem. Since the lengths of these sides are, respectively, $\sqrt{R}$ and R, it is clear that the integral of $f(s)$ along these three sides goes to 0 as $R \to \infty$. Meanwhile, the integral along the lower horizontal side tends to the integral of $f(s)$ along the whole real line. Now $f(s)$ has only one pole in the upper half plane, at $s = i\mu$, and so we see that

$$\int_{-\infty}^{\infty} f(s)\, ds = 2\pi i \operatorname*{Res}_{s=i\mu} f(s) = 2\pi i \frac{i\mu e^{-\mu|x|}}{2i\mu} = \pi i e^{-\mu|x|}.$$

Therefore, from the discussion in the problem statement, we see that

$$(2\pi)^{-3/2} \check{f} = \frac{1}{(2\pi)^2 i|x|} \pi i e^{-\mu|x|} = \frac{1}{4\pi} \frac{e^{-\mu|x|}}{|x|}.$$

In order to express $[(H_0 + \mu^2)^{-1}\varphi](x)$ as an integral, notice that we have

$$[(H_0 + \mu^2)(\check{f} * \varphi)]\hat{\ } = (\xi^2 + \mu^2)(\widehat{\check{f} * \varphi}) = (\xi^2 + \mu^2)f \cdot \hat{\varphi} = \hat{\varphi},$$

since $f = \frac{1}{\xi^2 + \mu^2}$. We see therefore that

$$(H_0 + \mu^2)(\check{f} * \varphi) = \varphi,$$

and so $[(H_0 + \mu^2)^{-1}\varphi](x) = \check{f} * \varphi$. In the last paragraph we computed $\check{f} = \frac{1}{4\pi} \frac{e^{-\mu|x|}}{|x|}$, and we see from this that its convolution with φ is

$$[(H_0 + \mu^2)^{-1}\varphi](x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{e^{-\mu|x-y|}}{|x-y|} \varphi(y)\, dy,$$

as desired.