Completeness and completion

Math 212

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1 Metric spaces

A **metric** for a set X is a function d from X to the real numbers $\mathbf{R}$,

$$d : X \times X \to \mathbf{R}$$

such that for all $x, y, z \in X$

1. $d(x, y) = d(y, x)$
2. $d(x, z) \leq d(x, y) + d(y, z)$
3. $d(x, x) = 0$
4. If $d(x, y) = 0$ then $x = y$.

The inequality in 2) is known as the **triangle inequality** since if X is the plane and d the usual notion of distance, it says that the length of an edge of a triangle is at most the sum of the lengths of the two other edges. (In the plane, the inequality is strict unless the three points lie on a line.)

Condition 4) is in many ways inessential, and it is often convenient to drop it, especially for the purposes of some proofs. For example, we might want to consider the decimal expansions $.49999\ldots$ and $.50000\ldots$ as different, but as having zero distance from one another. Or we might want to "identify" these two decimal expansions as representing the same point.

A function d which satisfies only conditions 1) - 3) is called a **pseudo-metric**.

A **metric space** is a pair (X, d) where X is a set and d is a metric on X. Almost always, when d is understood, we engage in the abuse of language and speak of "the metric space X".

Similarly for the notion of a **pseudo-metric space**.

In like fashion, we call $d(x, y)$ the **distance** between x and y, the function d being understood.

If r is a positive number and $x \in X$, the (open) **ball of radius** r about x is defined to be the set of points at distance less than r from x and is denoted by $B_r(x)$. In symbols,

$$B_r(x) := \{y|\ d(x, y) < r\}.$$

If r and s are positive real numbers and if x and z are points of a pseudo-metric space X, it is possible that $B_r(x) \cap B_s(z) = \emptyset$. This will certainly be the case if $d(x, z) > r + s$ by virtue of the triangle inequality. Suppose that this intersection is not empty and that

$$w \in B_r(x) \cap B_s(z).$$

If $y \in X$ is such that $d(y, w) < \min[r - d(x, w), s - d(z, w)]$ then the triangle inequality implies that $y \in B_r(x) \cap B_s(z)$. Put another way, if we set $t := \min[r - d(x, w), s - d(z, w)]$ then

$$B_t(w) \subset B_r(x) \cap B_s(z).$$

Put still another way, this says that the intersection of two (open) balls is either empty or is a union of open balls. So if we call a set in X **open** if either it is empty, or is a union of open balls, we conclude that the intersection of any finite number of open sets is open, as is the union of any number of open sets. In technical language, we say that the open balls form a base for a topology on X.

A map $f : X \to Y$ from one pseudo-metric space to another is called **continuous** if the inverse image under f of any open set in Y is an open set in X. Since an open set is a union of balls, this amounts to the condition that the inverse image of an open ball in Y is a union of open balls in X, or, to use the familiar ϵ, δ language, that if $f(x) = y$ then for every $\epsilon > 0$ there exists a $\delta = \delta(x, \epsilon) > 0$ such that

$$f(B_\delta(x)) \subset B_\epsilon(y).$$

Notice that in this definition δ is allowed to depend both on x and on ϵ. The map is called uniformly continuous if we can choose the δ independently of x.

An even stronger condition on a map from one pseudo-metric space to another is the **Lipschitz condition**. A map $f : X \to Y$ from a pseudo-metric space (X, d_X) to a pseudo-metric space (Y, d_Y) is called a **Lipschitz map** with **Lipschitz constant** C if

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2) \quad \forall x_1, x_2 \in X.$$

Clearly a Lipschitz map is uniformly continuous.

For example, suppose that A is a fixed subset of a pseudo-metric space X. Define the function $d(A, \cdot)$ from X to $\mathbf{R}$ by

$$d(A, x) := \inf\{d(x, w),\ w \in A\}.$$

The triangle inequality says that

$$d(x, w) \leq d(x, y) + d(y, w)$$

for all w, in particular for $w \in A$, and hence taking lower bounds we conclude that

$$d(A, x) \leq d(x, y) + d(A, y).$$

or

$$d(A, x) - d(A, y) \leq d(x, y).$$

Reversing the roles of x and y then gives

$$|d(A, x) - d(A, y)| \leq d(x, y).$$

Using the standard metric on the real numbers where the distance between a and b is $|a - b|$ this last inequality says that $d(A, \cdot)$ is a Lipschitz map from X to $\mathbf{R}$ with $C = 1$.

A closed set is defined to be a set whose complement is open. Since the inverse image of the complement of a set (under a map f) is the complement of the inverse image, we conclude that the inverse image of a closed set under a continuous map is again closed.

For example, the set consisting of a single point in $\mathbf{R}$ is closed. Since the map $d(A, \cdot)$ is continuous, we conclude that the set

$$\{x|d(A, x) = 0\}$$

consisting of all point at zero distance from A is a closed set. It clearly is a closed set which contains A. Suppose that S is some closed set containing A, and $y \notin S$. Then there is some $r > 0$ such that $B_r(y)$ is contained in the complement of S, which implies that $d(y, w) \geq r$ for all $w \in S$. Thus $\{x|d(A, x) = 0\} \subset S$. In short $\{x|d(A, x) = 0\}$ is a closed set containing A which is contained in all closed sets containing A. This is the definition of the **closure** of a set, which is denoted by $\overline{A}$. We have proved that

$$\overline{A} = \{x|d(A, x) = 0\}.$$

In particular, the closure of the one point set $\{x\}$ consists of all points u such that $d(u, x) = 0$.

Now the relation $d(x, y) = 0$ is an equivalence relation, call it R. (Transitivity being a consequence of the triangle inequality.) This then divides the space X into equivalence classes, where each equivalence class is of the form $\overline{\{x\}}$, the closure of a one point set. If $u \in \overline{\{x\}}$ and $v \in \overline{\{y\}}$ then

$$d(u, v) \leq d(u, x) + d(x, y) + d(y, v) = d(x, y).$$

since $x \in \overline{\{u\}}$ and $y \in \overline{\{v\}}$ we obtain the reverse inequality, and so

$$d(u, v) = d(x, y).$$

In other words, we may define the distance function on the quotient space X/R, i.e. on the space of equivalence classes by

$$d(\overline{\{x\}}, \overline{\{y\}}) := d(u,v), \quad u \in \overline{\{x\}}, v \in \overline{\{y\}}$$

and this does not depend on the choice of u and v. Axioms 1)-3) for a metric space continue to hold, but now

$$d(\overline{\{x\}}, \overline{\{y\}}) = 0 \Rightarrow \overline{\{x\}} = \overline{\{y\}}.$$

In other words, X/R is a *metric* space. Clearly the projection map $x \mapsto \overline{\{x\}}$ is an isometry of X onto X/R. (An **isometry** is a map which preserves distances.) In particular it is continuous. It is also open.

In short, we have provided a canonical way of passing (via an isometry) from a pseudo-metric space to a metric space by identifying points which are at zero distance from one another.

A subset A of a pseudo-metric space X is called *dense* if its closure is the whole space. From the above construction, the image A/R of A in the quotient space X/R is again dense. We will use this fact in the next section in the following form:

If $f : Y \to X$ is an isometry of Y such that $f(Y)$ is a dense set of X, then f descends to a map F of Y onto a dense set in the metric space X/R.

2 Completeness and completion.

The usual notion of convergence and Cauchy sequence go over unchanged to metric spaces or pseudo-metric spaces Y. A sequence $\{y_n\}$ is said to **converge** to the point y if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that

$$d(y_n, y) < \epsilon \quad \forall\, n > N.$$

A sequence $\{y_n\}$ is said to be **Cauchy** if for any $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that

$$d(y_n, y_m) < \epsilon \quad \forall\, m, n > N.$$

The triangle inequality implies that every convergent sequence is Cauchy. But not every Cauchy sequence is convergent. For example, we can have a sequence of rational numbers which converge to an irrational number, as in the approximation to the square root of 2. So if we look at the set of rational numbers as a metric space R in its own right, not every Cauchy sequence of rational numbers converges in R. We must "complete" the rational numbers to obtain $\mathbf{R}$, the set of real numbers. We want to discuss this phenomenon in general.

So we say that a (pseudo-)metric space is **complete** if every Cauchy sequence converges. The key result of this section is that we can always "complete" a metric or pseudo-metric space. More precisely, we claim that

Any metric (or pseudo-metric) space can be mapped by a one to one isometry onto a dense subset of a complete metric (or pseudo-metric) space.

By the italicized statement of the preceding section, it is enough to prove this for a pseudo-metric spaces X. Let X_{seq} denote the set of Cauchy sequences in X, and define the distance between the Cauchy sequences $\{x_n\}$ and $\{y_n\}$ to be

$$d(\{x_n\}, \{y_n\}) := \lim_{n \to \infty} d(x_n, y_n).$$

It is easy to check that d defines a pseudo-metric on X_{seq}. Let $f : X \to X_{seq}$ be the map sending x to the sequence all of whose elements are x;

$$f(x) = (x, x, x, x, \cdots).$$

It is clear that f is one to one and is an isometry. The image is dense since by definition

$$\lim d(f(x_n), \{x_n\}) = 0.$$

Now since $f(X)$ is dense in X_{seq}, it suffices to show that any Cauchy sequence of points of the form $f(x_n)$ converges to a limit. But such a sequence converges to the element $\{x_n\}$. QED

Of special interest are vector spaces which have metric which is compatible with the vector space properties and which is complete: Let V be a vector space over the real or complex numbers. A **norm** is a real valued function

$$v \mapsto \|v\|$$

on V which satisfies

1. $\|v\| \geq 0$ and > 0 if $v \neq 0$,
2. $\|cv\| = |c|\|v\|$ for any real (or complex) number c, and
3. $\|v + w\| \leq \|v\| + \|w\| \ \forall \ \ v, w \in V$.

Then $d(v, w) := \|v-w\|$ is a metric on V, which satisfies $d(v+u, w+u) = d(v, w)$ for all $v, w, u \in V$. The ball of radius r about the origin is then the set of all v such that $\|v\| < r$. A vector space equipped with a norm is called a **normed vector space** and if it is complete relative to the metric it is called a **Banach space**.

Our construction shows that any vector space with a norm can be completed so that it becomes a Banach space.