Fejer's Theorem

Math212

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For a bounded continuous function f we write $C(f,n,x)$ for the n-th Cesaro sum of its Fourier series. So if

$$a_n = a_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

then

$$C(f,0,x) = a_0, \quad C(f,1,x) = \frac{1}{2}(a_{-1}e^{-ix} + 2a_0 + a_1e^{ix}),$$

$$C(f,2,x) = \frac{1}{3}(a_{-2}e^{-2ix} + 2a_{-1}e^{-ix} + 3a_0 + 2a_1e^{ix} + a_2e^{2ix})$$

and, in general

$$\begin{aligned}
C(f,n,x) &= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} a_r e^{irx} \\
&= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} e^{-irt}dt\right) e^{irx} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{ir(x-t)}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_n(x-t)dt, \quad \text{where} \\
K_n(s) &:= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{ins} \quad \text{so setting } y = x-t \\
C(f,n,x) &= \frac{1}{2\pi}\int f(x-y)K_n(y)dy
\end{aligned}$$

where we have used the periodicity of f and K_n.

We have

$$(e^{-is/2} + e^{is/2})^2 = e^{-is} + 2 + e^{is} = 2K_1(s),$$

$$(e^{-is} + 1 + e^{is})^2 = e^{-2is} + 2e^{-is} + 3 + 2e^{is} + e^{2is} = 3K_2(s),$$

and, in general

$$(n+1)K_n(s) = \left(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})s}\right)^2 = \left(e^{-\frac{ins}{2}} \sum_{k=0}^{n} e^{iks}\right)^2.$$

If $s \neq 0 \bmod 2\pi$ we can sum this last sum as a geometric series so $(n+1)K_n(s) =$

$$\left(e^{-\frac{ins}{2}} \frac{1-e^{i(n+1)s}}{1-e^{is}}\right)^2 = \left(\frac{e^{-\frac{i(n+1)s}{2}} - e^{\frac{i(n+1)s}{2}}}{e^{-\frac{is}{2}} - e^{\frac{is}{2}}}\right)^2$$

so

$$K_n(s) = \frac{1}{n+1}\left(\frac{\sin\frac{(n+1)s}{2}}{\sin\frac{s}{2}}\right)^2$$

for $s \neq 0 \pmod{2\pi}$ while continuity gives

$$K_n(0) = n+1.$$

Below are some graphs of the Fejer kernels F_n:
Notice that

$$K_n(s) \geq 0 \quad \text{for all} \quad s$$

and from its original definition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s)ds = 1$$

since all the exponential terms integrate to zero.

We claim that for any $\delta > 0$ and any $\epsilon > 0$ there is an $N = N(\delta, \epsilon)$ such that

$$K_n(s) < \epsilon \;\text{ if } |s - 2\pi r| > \delta \;\text{ for all integers } r \;\text{ and } n > N.$$

Indeed, on this range, $|\sin(s/2)| > |\sin(\delta/2)| > 0$ so the denominator in the expression for K_n is bounded from below, while the numerator is bounded by 1.

Now suppose that f is periodic and continuous, and let M be such that

$$|f(s)| < M$$

for all s. Then f is uniformly continuous, so, for any $\epsilon > 0$ we can find a $\delta > 0$ so that

$$|f(s) - f(t)| < \epsilon/2 \;\text{ if } |s - t| < \delta.$$

We can then find an N such that

$$K_n(s) < \frac{\epsilon}{4M} \;\text{ if } |s - 2\pi r| > \delta \;\text{ and } n > N.$$

Then for $n > N$ we have

$$|C(f,n,t) - f(t)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)K_n(s)ds - \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s)f(t)ds\right| =$$

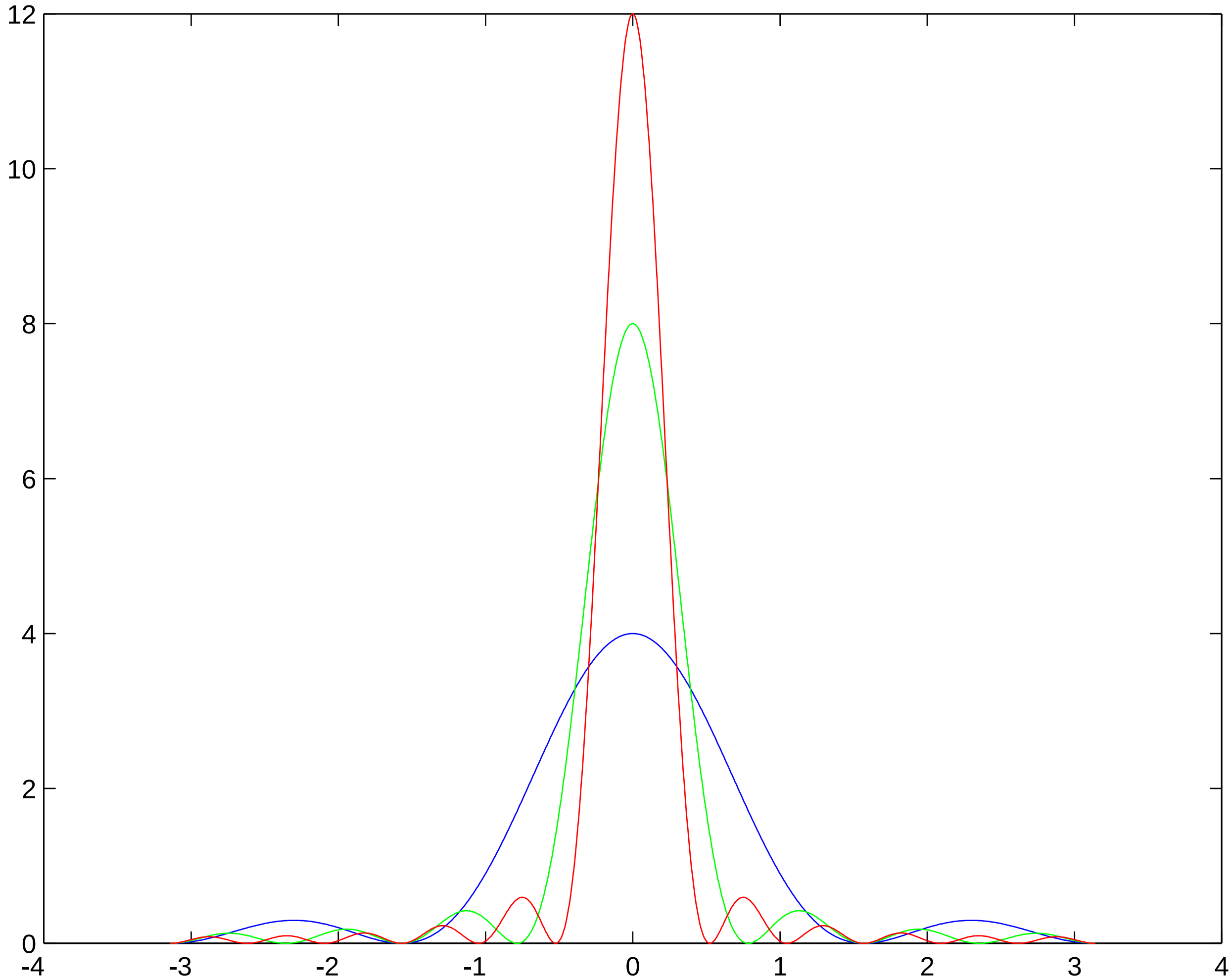

Figure 1: The graphs of K_3, K_7 and K_{11} over $[-\pi, \pi]$.

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(t-s) - f(t)] K_n(s) ds \right| .$$

We can break this integral up into two parts. The first, over the interval $s \in [-\delta, \delta]$ is at most

$$\frac{\epsilon}{2} \frac{1}{2\pi} \int_{[-\delta.\delta]} K_n(s) ds \leq \frac{\epsilon}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(s) ds = \frac{\epsilon}{2}$$

(since K_n is non-negative) while the second is at most

$$\frac{2M}{2\pi} \int_{s \notin [-\delta,\delta]} \frac{\epsilon}{4M} ds \leq \frac{\epsilon}{2}.$$

So

$$|C(f,n,t) - f(t)| \leq \epsilon \quad \text{if } n > N.$$

Thus we have proved Fejer's theorem which asserts that the Cesaro sum $C(f,n,t)$ of a continuous periodic function approaches $f(t)$ uniformly.

Thus

- The trigonometic polynomials are dense in the space of continuous periodic functions in the uniform topology.
- If f and g are continuous and period and have the same Fourier coefficients then they are equal.
- The Weierstrass approximation theorem: Any continuous function on a compact interval can be uniformly approximated by polynomials. (Since we can extend the function to be periodic, then approximate the extended function by trigonometric polynmials, and then use the Taylor series of each exponential to approximate by polynomials.)