

Gibbs phenomenon

Math 212

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The “square wave” is defined as

$$s(x) := \begin{cases} -1, & -\pi < t < 0, \\ 1, & 0 < t < \pi \end{cases}$$

Its Fourier series can be written as

$$\frac{4}{\pi} \sum_{k \geq 1}^{\infty} \frac{\sin(2k-1)t}{2k-1}.$$

Indeed,

$$a_n = \frac{1}{2\pi} \left[\int_0^\pi e^{-inx} dx - \int_{-\pi}^0 e^{-inx} dx \right] = \frac{1}{2\pi} \frac{2}{-in} [e^{-in\pi} - 1].$$

The last expression in brackets vanishes when n is even, and equals -2 if n is odd. So the odd terms of the Fourier series vanish, while the sum of the terms involving $e^{\pm inx}$ for odd n is

$$\frac{1}{2\pi} \frac{4}{i|n|} [e^{inx} - e^{-inx}] = \frac{4}{\pi|n|} \sin |n|x.$$

We let s_n denote the n -th partial sum. Here are the graphs of s_n for some values of n :

Notice that there is blip overshooting the square wave whose distance from the horizontal (of about .18) does not decrease as n increases. Rather the width of the blip appears to go to zero. Let us verify these facts.

To locate the maxima we compute

$$\frac{ds_n}{dx} = \frac{4}{\pi} \sum_{k=1}^n \cos(2k-1)t$$

which we can sum via a geometric series as

$$\frac{4}{\pi} \frac{1}{2} \left(e^{ix} \sum_0^{n-1} e^{2kix} + e^{-ix} \sum_0^{n-1} e^{-2kix} \right) = \frac{4}{\pi} \frac{1}{2} \left(e^{ix} \frac{1 - e^{2inx}}{1 - e^{2ix}} + e^{-ix} \frac{1 - e^{-2inx}}{1 - e^{-2ix}} \right) =$$

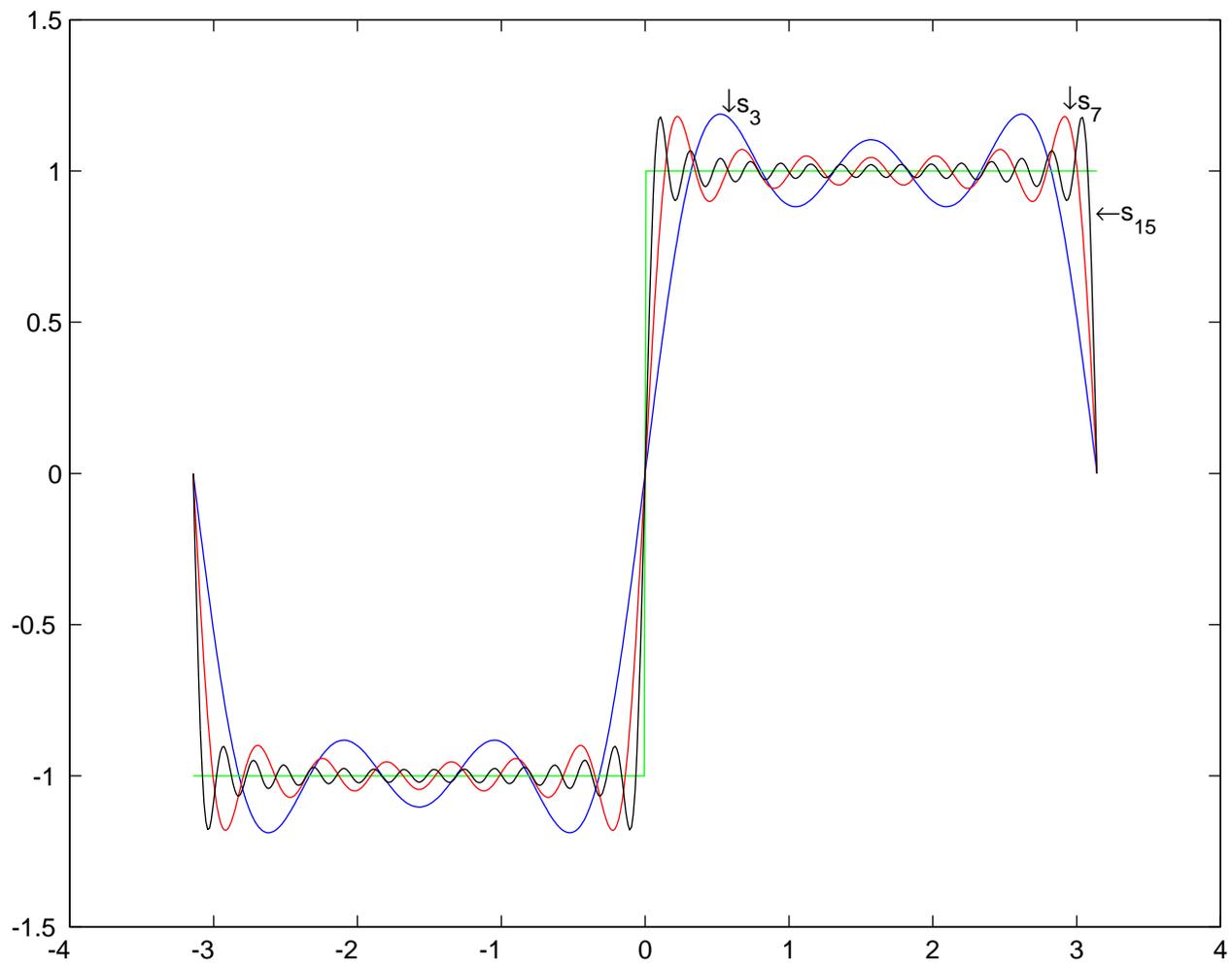


Figure 1: The graphs of s and s_3 , s_7 and s_{15} over $[-\pi, \pi]$.

$$\frac{2 \sin 2nx}{\pi \sin x}.$$

This extends by continuity to all x . This function vanishes at

$$x = \pm \frac{\pi}{2n}.$$

So these are the extrema nearest to the origin. Differentiating one more time gives $\sin(2n+1)x/\sin^2 x$ so we have a maximum to the right and a minimum to the left of the origin. Evaluating s_n at its maximum near the origin gives

$$s_n\left(\frac{\pi}{2n}\right) = \frac{4}{\pi} \sum_{k=1}^n \frac{\sin[(2k-1)\frac{\pi}{2n}]}{2k-1} = \frac{2}{\pi} \sum \frac{\sin[(2k-1)\frac{\pi}{2n}]}{(2k-1)\frac{\pi}{2n}} \cdot \frac{\pi}{n} \rightarrow \frac{2}{\pi} \int_0^\pi \frac{\sin t}{t} dt$$

as a Riemann integral. Evaluating of the integral gives approximately 1.18. Indeed, we have the power series expansion

$$\frac{\sin t}{t} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r+1)!}$$

which is valid for all t and we may integrate term by term to get

$$\int_0^\pi \frac{\sin t}{t} dt = \pi \left(1 - \frac{\pi^2}{3!3} + \frac{\pi^4}{5!5} - \frac{\pi^6}{7!7} + \dots \right).$$

As the series is an oscillating series with decreasing terms, the error involved in truncating the series is at most the first term neglected, and stopping at an odd term overshoots the mark while stopping at an even term undershoots the mark. Summing the first five terms gives 1.179384 while adding the next term gives 1.178957.

So this overshoot never disappears. This is the **Gibbs phenomenon**, Gibbs, 1899, first discovered by Wilbraham in 1848 and proved as a general phenomenon - that the overshoot at the jump is about 9% of the total jump by Bocher in 1906.

It appears from the figure that we have uniform convergence to the square wave as long as we stay a positive distance away from the jump. This is an illustration of Dirichlet's theorem. We also know that the square wave belongs to L_2 and that its Fourier series converges to it in the L_2 norm. So the Gibbs phenomenon shows something about the subtlety of L_2 convergence.

The human eye is sensitive to discontinuities, and unless corrected for, the Gibbs phenomenon shows up as "ghosts" on a TV screen.