

Summary of Lec.1

Math 212a

Sept. 21, 2000

- Because mathematicians at the beginning of the 19th century were unhappy with formulas such as

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

(see *phonysums*) they were concerned about how to interpret the equation

$$f(x) = \sum a_n e^{inx}, \quad a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

For example for the “square wave”

$$s(x) := \begin{cases} -1, & -\pi < t < 0, \\ 1, & 0 < t < \pi \end{cases}$$

the series is

$$\frac{4}{\pi} \sum_{n \geq 1}^{\infty} \frac{\sin(2n-1)t}{2n-1}.$$

How could this series converge when the series of its coefficients

$$\frac{4}{\pi} \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \right)$$

diverges?

- Dirichlet proved in 1829 that the Fourier series of any piecewise differentiable function converges at all points, to $f(x)$ at points of continuity of f , and to the average of the limiting values, $\frac{1}{2}[f(x+) + f(x_-)]$ at points of discontinuity. Thus the Fourier series of the square converges at all points.
- However at points of discontinuity we have the *Gibbsphenomenon*: There is a blip with height about 9% of the jump which never disappears in the approximations, the blip just gets narrower and narrower.
- Fejer proved in 1900 that the Cesaro means of the Fourier series of a piecewise continuous function always converge. See *Fejer*. Any continuous function on \mathbf{T} can be uniformly approximated by trigonometrical series.

- As a consequence any continuous function on a closed bounded interval can be uniformly approximated by polynomials. This is the Weierstrass approximation theorem.
- A subset $A \subset \mathbf{R}$ is said to have measure zero if, for every $\epsilon > 0$ we can find countable many intervals I_i such that

$$A \subset \bigcup I_i \quad \text{and} \quad \sum |I_i| < \epsilon$$

where $|I|$ denotes the length of the interval I .

- The union of a countable collection of sets of measure zero has measure zero. In particular, the set of rational numbers has measure zero.