Summary of Lec.3

Math 212a

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All details can be found in *hilbertspace* in the folder for Lecture 1

- **Bessel's inequality** asserts that if $\{\phi_i\}$ is an orthonormal set of vectors and v an arbitrary vector in a Hilbert space, then

$$\sum |(v,\phi_i|^2 leq \|v\|^2.$$

We get equality for all v in the above equation if and only if the set of finite linear combinations of the ϕ_i is dense. We know this to be true for the orthonormal set $\{e^{ikx}\}$ in $L_2(\mathbf{T})$ by Fejer's theorem.

- **Self adjoint operators.** These are the ones which satisfy $(Tu,v) = (u,Tv)$ for all u,v in the Hilbert space. In particular (Tv,v) is real and any eigenvalue of T is real.

- **Non-negative operators** are those self adjoint operators which satisfy $(Tv,v) \geq 0$ for all v. If T is a non-negative operator, then

$$\|Tv\| \leq \|T\|^{\frac{1}{2}}(Tv,v)^{\frac{1}{2}}$$

for all v. In particular $(Tv_n,v_n) \to 0$ implies that $Tv_n \to 0$. For any self adjoint operator T, the operator $\|T\|I - T$ is non-negative.

- **Compact operators**. These satisfy the following condition: If v_n is any sequence of vectors with $\|v_n\| = 1$ for all n, then we can find a subsequence such that Tv_{n_j} converges. The main theorem about compact self adjoint operators is:

Let T be a compact self-adjoint operator. Then $R(T)$ has an orthonormal basis $\{\phi_i\}$ consisting of eigenvectors of T and if $R(T)$ is infinite dimensional then the corresponding sequence $\{r_n\}$ of eigenvalues converges to 0.

The idea of the proof is to look at a sequence of unit vectors $\{u_n\}$ such that $\|Tu_n\|$ approaches the maximum value $m_1 := \|T\|$ of $\|Tu\|$ for u on the unit sphere, and choose a subsequence such that Tu_{n_j} converges. Then one shows that the limit w is an eigenvector of T^2 with eigenvalue

$\|T\|^2$. Either w is an eigenvector of T with eigenvalue m_1 or $(T - m_1)w$ is an eigenvector of T with eigenvalue $-m_1$. One then aplies the procedure over again to the subspace $w^\perp$ and repeats.