Fourier Transform

Math 212a

September 28, 2000

Contents

1 Conventions, especially about 2π.

The space $\mathcal{S}$ consists of all functions on $\mathbf{R}$ which are infinitely differentiable and vanish at infinity rapidly with all their derivatives in the sense that

$$\|f\|_{m,n} := \sup\{|x^m f^{(n)}(x)|\} < \infty.$$

The $\|\cdot\|_{m,n}$ give a family of semi-norms on $\mathcal{S}$ making $\mathcal{S}$ into a Frechet space - that is, a vector space space whose topology is determined by a countble family of semi-norms. More about this later in the course We use the measure

$$\frac{1}{\sqrt{2\pi}}dx$$

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on $\mathbf{R}$ and so define the Fourier transform of an element of $\mathcal{S}$ by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-ix\xi} dx$$

and the convolution of two elements of $\mathcal{S}$ by

$$(f \star g)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x-t) g(t) dt.$$

The Fourier transform is well defined on $\mathcal{S}$ and

$$\left[\left(\frac{d}{dx} \right)^m ((-ix)^n f) \right]\hat{} = (-i\xi)^m \left(\frac{d}{d\xi} \right) \hat{f},$$

as follows by integration by parts. This show that the Fourier transform maps $\mathcal{S}$ to $\mathcal{S}$.

2   Convolution goes to multiplication.

$$
\begin{aligned}
(f \star g)\hat{}(\xi) &= \frac{1}{2\pi} \int \int f(x-t) g(t) dt e^{-ix\xi} dx \\
&= \frac{1}{2\pi} \int \int f(u) g(t) e^{-i(u+t)\xi} du dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(u) e^{-iu\xi} du \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} g(t) e^{-it\xi} dt
\end{aligned}
$$

so

$$(f \star g)\hat{} = \hat{f}\hat{g}.$$

3   Scaling.

For any $f \in \mathcal{S}$ and $a > 0$ define $S_a f$ by $(S_a)f(x) := f(ax)$. Then setting $u = ax$ so $dx = (1/a)du$ we have

$$
\begin{aligned}
(S_a f)\hat{}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(ax) e^{-ix\xi} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (1/a) f(u) e^{-iu(\xi/a)} du
\end{aligned}
$$

so

$$(S_a f)\hat{} = (1/a) S_{1/a} \hat{f}.$$

4 Fourier transform of a Gaussian is a Gaussian.

The polar coordinate trick evaluates

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-x^2/2} dx = 1.$$

The integral

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-x^2/2 - x\eta} dx$$

converges for all complex values of η, uniformly in any compact region. Hence it defines an analytic function of η that can be evaluated by taking η to be real and then using analytic continuation. For real η we complete the square and make a change of variables:

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-x^2/2 - x\eta} dx &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-(x+\eta)^2/2 + \eta^2/2} dx \\
&= e^{\eta^2/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-(x+\eta)^2/2} dx \\
&= e^{\eta^2/2}.
\end{aligned}
$$

Setting $\eta = i\xi$ gives

$$\hat{n} = n \quad \text{if } n(x) := e^{-x^2/2}.$$

If we set $a = \epsilon\sqrt{2}$ in our scaling equation and define

$$\rho_\epsilon := n_{\epsilon\sqrt{2}}$$

so

$$\rho_\epsilon(x) = e^{-\epsilon^2 x^2},$$

then

$$(\rho_\epsilon)\hat{}(x) = \frac{1}{\epsilon\sqrt{2}} e^{-x^2/4\epsilon^2}.$$

Notice that for any $g \in \mathcal{S}$ we have

$$\int_{\mathbf{R}} (1/a)(S_a g)(\xi) d\xi = \int_{\mathbf{R}} g(\xi) d\xi$$

so setting $a = \epsilon\sqrt{2}$ we conclude that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (\rho_\epsilon)\hat{}(\xi) d\xi = 1$$

for all ϵ.

Let

$$\psi := \psi_1 := (\rho_1)\hat{}$$

and

$$\psi_\epsilon := (\rho_\epsilon)\hat{}.$$

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Then

$$(\psi_\epsilon \star g)(\xi) - g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} [g(\xi - \eta) - g(\xi)] \frac{1}{\epsilon} \psi\left(\frac{\eta}{\epsilon}\right) d\eta =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} [g(\xi - \epsilon\zeta) - g(\xi)] \psi(\zeta) d\zeta.$$

Since $g \in \mathcal{S}$ it is uniformly continuous on $\mathbf{R}$, so that for any $\delta > 0$ we can find ϵ_0 so that the above integral is less than δ in absolute value for all $0 < \epsilon < \epsilon_0$. In short,

$$\|\psi_\epsilon \star g - g\|_\infty \to 0, \quad \text{as } \epsilon \to 0.$$

5   The multiplication formula.

This says that

$$\int_{\mathbf{R}} \hat{f}(x) g(x) dx = \int_{\mathbf{R}} f(x) \hat{g}(x) dx$$

for any $f, g \in \mathcal{S}$. Indeed the left hand side equals

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x) e^{-ixy} dy g(x) dx.$$

We can write this integral as a double integral and then interchange the order of integration which gives the right hand side.

6   The inversion formula.

This says that for any $f \in \mathcal{S}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(\xi) e^{ix\xi} d\xi.$$

To prove this, we first observe that for any $h \in \mathcal{S}$ the Fourier transform of $x \mapsto e^{i\eta x} h(x)$ is just $\xi \mapsto \hat{h}(\xi - \eta)$ as follows directly from the definition.

Taking $g(x) = e^{i\eta x} e^{-\epsilon^2 x^2}$ in the multiplication formula gives

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(t) e^{itx} e^{-\epsilon^2 t^2} dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(t) \psi_\epsilon(t - x) dt = (f \star \psi_\epsilon)(x).$$

We know that the right hand side approaches $f(x)$ as $\epsilon \to 0$. Also, $e^{-\epsilon^2 t^2} \to 1$ for each fixed t, and in fact uniformly on any bounded t interval. Also $0 < e^{-\epsilon^2 t^2} \le 1$ for all t. So choosing the interval of integration large enough, we can take the left hand side as close as we like to $\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{f}(x) e^{ixt} dt$ by then choosing ϵ sufficiently small. QED

7 Plancherel's theorem

Let
$$\tilde{f}(x) := \overline{f(-x)}.$$
Then the Fourier transform of $\tilde{f}$ is given by
$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \overline{f(-x)} e^{-ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \overline{f(u)} e^{iu\xi} du$$
so
$$(\tilde{f})\hat{} = \overline{\hat{f}}.$$
Thus
$$(f \star \tilde{f})\hat{} = |\hat{f}|^2.$$
The inversion formula applied to $f \star \tilde{f}$ and evaluated at 0 gives
$$(f \star \tilde{f})(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} |\hat{f}|^2 dx.$$

The left hand side of this equation is
$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x)\tilde{f}(0-x) dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} |f(x)|^2 dx.$$

Thus we have proved **Plancherel's formula**
$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} |f(x)|^2 dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} |\hat{f}(x)|^2 dx.$$

Since $\mathcal{S}$ is dense in $L_2(\mathbf{R})$ we conclude that the Fourier transform extends to unitary isomorphism of $L_2(\mathbf{R})$ onto itself.

8 The Poisson summation formula.

This says that for any $g \in \mathcal{S}$ we have
$$\sum_k g(2\pi k) = \frac{1}{\sqrt{2\pi}} \sum_m \hat{g}(m).$$

To prove this let
$$h(x) := \sum_k g(x + 2\pi k)$$
so h is a smooth function, periodic of period 2π and
$$h(0) = \sum_k g(2\pi k).$$

We may expand h into a Fourier series

$$h(x) = \sum_m a_m e^{imx}$$

where

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} h(x)e^{-imx}\,dx = \frac{1}{2\pi} \int_{\mathbf{R}} g(x)e^{-imx}\,dx = \frac{1}{\sqrt{2\pi}}\hat{g}(m).$$

Setting $x = 0$ in the Fourier expansion

$$h(x) = \frac{1}{\sqrt{2\pi}} \sum \hat{g}(m)e^{imx}$$

gives

$$h(0) = \frac{1}{\sqrt{2\pi}} \sum_m \hat{g}(m).$$

9 The Shannon sampling theorem.

Let $f \in \mathcal{S}$ be such that its Fourier transform is supported in the interval $[-\pi, \pi]$. Then a knowledge of $f(n)$ for all $n \in \mathbf{Z}$ determines f. More explicitly,

$$f(t) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(n-t)}{n-t}. \tag{1}$$

Proof. Let g be the periodic function (of period 2π) which extends $\hat{f}$, the Fourier transform of f. So

$$g(\tau) = \hat{f}(\tau), \quad \tau \in [-\pi, \pi]$$

and is periodic.

Expand g into a Fourier series:

$$g = \sum_{n\in\mathbf{Z}} c_n e^{in\tau},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\tau)e^{-in\tau}\,d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\tau)e^{-in\tau}\,d\tau,$$

or

$$c_n = \frac{1}{(2\pi)^{\frac{1}{2}}}f(-n).$$

But

$$f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \hat{f}(\tau)e^{it\tau}\,d\tau = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\pi}^{\pi} g(\tau)e^{it\tau}\,d\tau =$$

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\pi}^{\pi} \sum \frac{1}{(2\pi)^{\frac{1}{2}}} f(-n) e^{i(n+t)\tau} \, d\tau.$$

Replacing n by $-n$ in the sum, and interchanging summation and integration, which is legitimate since the $f(n)$ decrease very fast, this becomes

$$f(t) = \frac{1}{2\pi} \sum_n f(n) \int_{-\pi}^{\pi} e^{i(t-n)\tau} \, d\tau.$$

But

$$\int_{-\pi}^{\pi} e^{(t-n)\tau} \, d\tau = \frac{e^{(t-n)\tau}}{i(t-n)} \bigg|_{-\pi}^{\pi} = \frac{e^{i(t-n)\pi} - e^{i(t-n)\pi}}{i(t-n)} = 2 \frac{\sin \pi(n-t)}{n-t}. \quad \text{QED}$$

It is useful to reformulate this via rescaling so that the interval $[-\pi, \pi]$ is replaced by an arbitrary interval symmetric about the origin: In the engineering literature the **frequency** λ is defined by

$$\xi = 2\pi\lambda.$$

Suppose we want to apply (1) to $g = S_a f$. We know that the Fourier transform of g is $(1/a)S_{1/a}\hat{f}$ and

$$\operatorname{supp} S_{1/a}\hat{f} = a \operatorname{supp} \hat{f}.$$

So if

$$\operatorname{supp} \hat{f} \subset [-2\pi\lambda_c, 2\pi\lambda_c]$$

we want to choose a so that $a 2\pi\lambda_c \leq \pi$ or

$$a \leq \frac{1}{2\lambda_c}. \tag{2}$$

For a in this range (1) says that

$$f(ax) = \frac{1}{\pi} \sum f(na) \frac{\sin \pi(x-n)}{x-n},$$

or setting $t = ax$,

$$f(t) = \sum_{n=-\infty}^{\infty} f(na) \frac{\sin \frac{\pi}{a}(t-na)}{\frac{\pi}{a}(t-na)}. \tag{3}$$

This holds in L_2 under the assumption that f satisfies $\operatorname{supp} \hat{f} \subset [-2\pi\lambda_c, 2\pi\lambda_c]$. We say that f has **finite bandwidth** or is **bandlimited** with bandlimit λ_c. The critical value $a_c = 1/2\lambda_c$ is known as the **Nyquist sampling interval** and $(1/a) = 2\lambda_c$ is known as the **Nyquist sampling rate**. Thus the Shannon sampling theorem says that a band-limited signal can be recovered completely from a set of samples taken at a rate $\geq$ the Nyquist sampling rate.

10 The Heisenberg Uncertainty Principle.

Suppose that $f \in \mathcal{S}(\mathbf{R})$ satisfies

$$\int |f|^2 = 1.$$

We can think of $|f|^2$ as giving a "probability density" and then would define the mean of this probability density as

$$x_m := \int_{\mathbf{R}} x|f(x)|^2 dx.$$

Similarly, we would define the variance as

$$\int_{\mathbf{R}} (x - x_m)^2 |f(x)|^2 dx.$$

The Heisenberg uncertainty principle asserts that the product of the variance of $|f|^2$ and that of $|\hat{f}|^2$ cannot get below a certain bound.

Let us first treat the case where the mean of $|f|^2$ and of $|\hat{f}|^2$ are both zero. Then

$$\left(\int |xf(x)|^2 dx\right) \left(\int |\xi\hat{f}(\xi)|^2 d\xi\right) \geq \frac{1}{4}.$$

Proof. Write $-i\xi f(\xi)$ as the Fourier transform of f' and use Plancherel to write the second integral as $\int |f'(x)|^2 dx$. Then Cauchy-Schwartz says that the left hand side is $\geq$ the square of

$$\int |xf(x)f'(x)| dx \geq \left|\int \mathrm{Re}(xf(x)\overline{f'(x)}) dx\right| = \frac{1}{2}\left|\int x(f(x)\overline{f'(x)} + \overline{f(x)}f'(x)) dx\right|$$

$$= \frac{1}{2}\left|\int x\frac{d}{dx}|f|^2 dx\right| = \frac{1}{2}\left|\int -|f|^2 dx\right| = \frac{1}{2}.$$

If x_m or ξ_m, the mean of $|\hat{f}|^2$ are not necessarily zero. the Heisenberg uncertainty principle says that

$$\left(\int |(x - x_m)f(x)|^2 dx\right) \left(\int |(\xi - \xi_m)\hat{f}(\xi)|^2 d\xi\right) \geq \frac{1}{4}.$$

where, we recall

$$\xi_m := \int \xi|\hat{f}|^2(\xi) d\xi.$$

The general case is reduced to the special case by replacing $f(x)$ by

$$f(x + x_m)e^{i\xi_m x}.$$