Compactness

Math 212a

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1 Compactness.

A topological space X is said to be **compact** if it has one (and hence the other) of the following equivalent properties:

- Every open cover has a finite subcover. In more detail: if $\{U_\alpha\}$ is a collection of open sets with

$$X \subset \bigcup_\alpha U_\alpha$$

then there are finitely many $\alpha_1, \ldots, \alpha_n$ such that

$$X \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

- If $\mathcal{F}$ is a family of closed sets such that

$$\bigcap_{F \in \mathcal{F}} = \emptyset$$

then a finite intersection of the F's are empty:

$$F_1 \cap \cdots \cap F_n = \emptyset.$$

2 Total Boundedness.

A metric space X is said to be **totally bounded** if for every $\epsilon > 0$ there are finitely many open balls of radius ϵ which cover X.

Theorem 1 *The following assertions are equivalent for a metric space:*

1. *X is compact.*
2. *Every sequence in X has a convergent subsequence.*
3. *X is totally bounded and complete.*

Proof that 1. $\Rightarrow$ 2. Let $\{y_i\}$ be a sequence of points in X. We first show that there is a point x with the property for every $\epsilon > 0$, the open ball of radius ϵ centered at x contains the points y_i for infinitely many i. Suppose not. Then there is some $\epsilon > 0$ such that the ball of radius ϵ around any point z contains only finitely many y_i. Since z is contained in the ball centered around itself, the set of such balls covers X. By compactness, finitely many of these balls cover X, and hence there are only finitely many i, a contradiction.

Now choose i_1 so that y_{i_1} is in the ball of radius $\frac{1}{2}$ centered at x. Then choose $i_2 > i_1$ so that y_{i_2} is in the ball of radius $\frac{1}{4}$ centered at x and keep going. We have constructed a subsequence so that the points y_{i_k} converge to x. Thus we have proved that 1) implies 2.

Proof that 2. $\Rightarrow$ 3. If $\{x_j\}$ is a Cauchy sequence in X, it has a convergent subsequence by hypothesis, and the limit of this subsequence is (by the triangle inequality) the limit of the original sequence. Hence X is complete. We must show that it is totally bounded. Given $\epsilon > 0$, pick a point $y_1 \in X$ and let $B_\epsilon(y_1)$ be open ball of radius ϵ about y_1. If $B_\epsilon(y_1) = X$ there is nothing further to prove. If not, pick a point $y_2 \in X - B_\epsilon(y_1)$ and let $B_\epsilon(y_2)$ be the ball of radius ϵ about y_2. If $B_\epsilon(y_1) \cup B_\epsilon(y_2) = X$ there is nothing to prove. If not, pick a point $y_3 \in X - (B_\epsilon(y_1) \cup B_\epsilon(y_2))$ etc. This procedure can not continue indefinitely, for then we will have constructed a sequence of points which are all at a mutual distance $\geq \epsilon$ from one another, and this sequence has no Cauchy subsequence.

Proof that 3. $\Rightarrow$ 2. Let $\{x_j\}$ be a sequence of points in X which we relabel as $\{x_{1j}\}$. Let $B_{1,\frac{1}{2}}, \ldots, B_{n_1,\frac{1}{2}}$ be a finite number of balls of radius $\frac{1}{2}$ which cover X. Our hypothesis 3. asserts that such a finite cover exists. Infinitely many of the j must be such that the x_{i_j} all lie in one of these balls. Relabel this subsequence as $\{x_{2,j}\}$. Cover X by finitely many balls of radius $\frac{1}{3}$. There must be infinitely many j such that all the $x_{2,j}$ lie in one of the balls. Relabel this subsequence as $\{x_{3,j}\}$. Continue. At the ith stage we have a subsequence $\{x_{i,j}\}$ of our original sequence (in fact of the preceding subsequence in the construction) all of whose points lie in a ball of radius $1/i$. Now consider the "diagonal" subsequence

$$x_{1,1},\ x_{2,2}, x_{3,3}, \ldots.$$

All the points from $x_{i,i}$ on lie in a fixed ball of radius $1/i$ so this is a Cauchy sequence. Since X is assumed to be complete, this subsequence of our original sequence is convergent.

We have shown that 2. and 3. are equivalent. The hard part of the proof consists in showing that these two conditions imply 1. For this it is useful to introduce some terminology:

3 Separability.

A metric space X is called **separable** if it has a countable subset $\{x_j\}$ of points which are dense. For example $\mathbf{R}$ is separable because the rationals are countable and dense. Similarly, $\mathbf{R}^n$ is separable because the points all of whose coordinates are rational form a countable dense subset.

Proposition 1 *Any subset Y of a separable metric space X is separable (in the induced metric).*

Proof. Let $\{x_j\}$ be a countable dense sequence in X. Consider the set of pairs (j, n) such that

$$B_{1/n}(x_j) \cap Y \neq \emptyset.$$

For each such (j, n) let $y_{j,n}$ be any point in this non-empty intersection. We claim that the countable set of points $y_{j,n}$ are dense in Y. Indeed, let y be any point of Y. Let n be any positive integer. We can find a point x_j such that $d(x_j, y) < 1/2n$ since the x_j are dense in X. But then $d(y, y_{j,n}) < 1/n$ by the triangle inequality. QED

Proposition 2 *Any totally bounded metric space X is separable.*

Proof. For each n let $\{x_{1,n}, \ldots, x_{i_n,n}\}$ be the centers of balls of radius $1/n$ (finite in number) which cover X. Put all of these together into one sequence which is clearly dense. QED

A **base** for the open sets in a topology on a space X is a collection $\mathcal{B}$ of open set such that every open set of X is the union of sets of $\mathcal{B}$

Proposition 3 *A family $\mathcal{B}$ is a base for the topology on X if and only if for every $x \in X$ and every open set U containing x there is a $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U$.*

Proof. If $\mathcal{B}$ is a base, then U is a union of members on $\mathcal{B}$ one of which must therefore contain x. Conversely, let U be an open subset of X. For each $x \in U$ there is a $V_x \subset U$ belonging to $\mathcal{B}$. The union of these over all $x \in U$ is contained in U and contains all the points of U, hence equals U. So $\mathcal{B}$ is a base. QED

4 Second Countability.

A topological space X is said to be **second countable** or to satisfy the **second axiom of countability** if it has a base $\mathcal{B}$ which is (finite or) countable.

Proposition 4 *A metric space X is second countable if and only if it is separable.*

Proof. Suppose X is separable with a countable dense set $\{x_i\}$. The open balls of radius $1/n$ about the x_i form a countable base: Indeed, if U is an open set and $x \in U$ then take n sufficiently large so that $B_{2/n}(x) \subset U$. Choose j so that $d(x_j, x) < 1/n$. Then $V := B_{1/n}(x_j)$ satisfies $x \in V \subset U$ so by Proposition 3 the set of balls $B_{1/n}(x_j)$ form a base and they constitute a countable set. Conversely, let $\mathcal{B}$ be a countable base, and choose a point $x_j \in U_j$ for each $U_j \in \mathcal{B}$. If x is any point of X, the ball of radius $\epsilon > 0$ about x includes some U_j and hence contains x_j. So the x_j form a countable dense set. QED

Proposition 5 **Lindelof's theorem.** *Suppose that the topological space X is second countable. Then every open cover has a countable subcover.*

Let $\mathcal{U}$ be a cover, not necessarily countable, and let $\mathcal{B}$ be a countable base. Let $\mathcal{C} \subset \mathcal{B}$ consist of those open sets V belonging to $\mathcal{B}$ which such that $V \subset U$ where $U \in \mathcal{U}$. By Proposition 3 these form a (countable) cover. For each $V \in \mathcal{C}$ choose a $U_V \in \mathcal{U}$ such that $V \subset U_V$. Then the $\{U_V\}_{V \in \mathcal{C}}$ forma countable subset of $\mathcal{U}$ which is a cover. QED

5 Conclusion of the proof of Theorem 1.

Suppose that condition 2. and 3. of the theorem hold for the metric space X. By Proposition 2, X is separable, and hence by Proposition 4, X is second countable. Hence by Proposition 5, every cover $\mathcal{U}$ has a countable subcover. So we must prove that if $U_1,\ U_2,\ U_3,\ \dots$ is a sequence of open sets which cover X, then $X = U_1 \cup U_2 \cup \cdots \cup U_m$ for some finite integer m. Suppose not. For each m choose $x_m \in X$ with $x_m \notin U_1 \cup \cdots \cup U_m$. By condition 2. of Theorem 1, we may choose a subsequence of the $\{x_j\}$ which converge to some point x. Since $U_1 \cup \cdots U_m$ is open, its complement is closed, and since $x_j \notin U_1 \cup \cdots \cup U_m$ for $j > m$ we conclude that $x \notin U_1 \cup \cdots \cup U_m$ for any m. This says that the $\{U_j\}$ do *not* cover X, a contradiction. QED

Putting the pieces together, we see that a closed bounded subset of $\mathbf{R}^m$ is compact. This is the famous Heine-Borel theorem. So Theorem 1 can be considered as a far reaching generalization of the Heine-Borel theorem.

6 Dini's lemma.

Let X be a metric space and let L denote the space of real valued continuous functions of compact support. So $f \in L$ means that f is continuous, and the closure of the set of all x for which $|f(x)| > 0$ is compact. Thus L is a real vector space, and $f \in L \to |f| \in L$. Thus if $f \in L$ and $g \in L$ then $f+g \in L$ and also max $(f,g) = \frac{1}{2}(f+g+|f-g|) \in L$ and min $(f,g) = \frac{1}{2}(f+g-|f-g|) \in L$.

For a sequence of elements in L (or more generally in any space of real valued functions) we write $f_n \downarrow 0$ to mean that the sequence of functions is monotone decreasing, and at each x we have $f_n(x) \to 0$.

Theorem 2 Dini's lemma. *If $f_n \in L$ and $f_n \downarrow 0$ then $\|f_n\|_\infty \to 0$. In other words, monotone decreasing convergence to 0 implies uniform convergence to zero for elements of L.*

Proof. Given $\epsilon > 0$, let $C_n = \{x | f_n(x) \geq \epsilon\}$. Then the C_n are compact, $C_n \supset C_{n+1}$ and $\bigcap_k C_k = \emptyset$. Hence a finite intersection is already empty, which means that $C_n = \emptyset$ for some n. This means that $\|f_n\|_\infty \leq \epsilon$ for some n, and hence, since the sequence is monotone decreasing, for all subsequent n. QED

7 The Lebesgue outer measure of an interval is its length.

For any subset $A \subset \mathbf{R}$ we define its **Lebesgue outer measure** by

$$m^*(A) := \inf \sum \ell(I_n) : \; I_n \text{ are intervals with } A \subset \bigcup I_n. \tag{1}$$

Here the length $\ell(I)$ of any interval $I = [a,b]$ is $b - a$ with the same definition for half open intervals $(a,b]$ of $[a,b)$ of open intervals. Of course if $a = -\infty$ and b is finite or $+\infty$, or if a is finite and $b = +\infty$ the length is infinite. So the infimum in (1) is taken over all covers of A by intervals. By the usual $\epsilon/2^n$ trick, i.e. by replacing each $I_j = [a_j, b_j]$ by $(a_j - \epsilon/2^{j+1}, b_j + \epsilon/2^{j+1})$ we may assume that the infimum is taken over open intervals. (Equally well, we could use half open intervals of the form $[a,b)$, for example.).

It is clear that if $A \subset B$ then $m^*(A) \leq m^*(B)$ since any cover of B by intervals is a cover of A. Also, if Z is any set of measure zero, then $m^*(A \cup Z) = m^*(A)$. In particular, $m^*(Z) = 0$ if Z has measure zero. Also, if $A = [a,b]$ is an interval, then we can cover it by itself, so

$$m^*([a,b]) \leq b - a,$$

and hence the same is true for $(a,b]$, $[a,b)$, or (a,b). If the interval is infinite, it clearly can not be covered by a set of intervals whose total length is finite, since if we lined them up with end points touching they could not cover an infinite interval. We still must prove that

$$m^*(I) = \ell(I) \tag{2}$$

if I is a finite interval. We may assume that $I = [c,d]$ is a closed interval by what we have already said, and that the minimization in (1) is with respect to a cover by open intervals. So what we must show is that if

$$[c,d] \subset \bigcup_i (a_i,b_i)$$

then

$$d-c \leq \sum_i (b_i - a_i).$$

We first apply Heine-Borel to replace the countable cover by a finite cover. (This only decreases the right hand side of preceding inequality.) So let n be the number of elements in the cover. We want to prove that if

$$[c,d] \subset \bigcap_{i=1}^{n} (a_i,b_i) \text{ then } d-c \leq \sum_{i=1}^{n} (b_i - a_i).$$

We shall do this by induction on n. If $n = 1$ then $a_1 < c$ and $b_1 > d$ so clearly $b_1 - a_1 > d - c$.

Suppose that $n \geq 2$ and we know the result for all covers (of all intervals $[c,d]$) with at most $n-1$ intervals in the cover. If some interval (a_i,b_i) is disjoint from $[c,d]$ we may eliminate it from the cover, and then we are in the case of $n-1$ intervals. So every (a_i,b_i) has non-empty intersection with $[c,d]$. Among the the intervals (a_i,b_i) there will be one for which a_i takes on the minimum possible value. By relabeling, we may assume that this is (a_1,b_1). Since c is covered, we must have $a_1 < c$. If $b_1 > d$ then (a_1,b_1) covers $[c,d]$ and there is nothing further to do. So assume $b_1 \leq d$. We must have $b_1 > c$ since $(a_1,b_1) \cap [c,d] \neq \emptyset$. Since $b_1 \in [c,d]$, at least one of the intervals (a_i,b_i), $i > 1$ contains the point b_1. By relabeling, we may assume that it is (a_2,b_2). But now we have a cover of $[c,d]$ by $n-1$ intervals:

$$[c,d] \subset (a_1,b_2) \cup \bigcup_{i=3}^{n} (a_i,b_i).$$

So by induction

$$d-c \leq (b_2 - a_1) + \sum_{i=3}^{n} (b_i - a_i).$$

But $b_2 - a_1 \leq (b_2 - a_2) + (b_1 - a_1)$ since $a_2 < b_1$. QED

8 Zorn's lemma and the axiom of choice.

In the first few sections we repeatedly used an argument which involved "choosing" this or that element of a set. That we can do so is an axiom known as

The axiom of choice. *If F is a function with domain D such that $F(x)$ is a non-empty set for every $x \in D$ there exists a function f with domain D such that $f(x) \in F(x)$ for every $x \in D$.*

It has been proved by Gödel that if mathematics is consistent without the axiom of choice (a big "if") then mathematics remains consistent with the axiom of choice added.

In fact, it will be convenient for us to take a slightly less intuitive axiom as out starting point:

Zorn's lemma. *Every partially ordered set A has a maximally linearly ordered subset. If every linearly ordered subset of A has an upper bound, then A contains a maximum element.*

The second assertion is a consequence of the first. For let B be a maximum linearly ordered subset of A, and x an upper bound for B. Then x is a maximum element of A, for if $y \succ x$ then we could add y to B to obtain a larger linearly ordered set. Thus there is no element in A which is strictly larger than x which is what we mean when we say that x is a maximum element.

. Zorn's lemma implies the axiom of choice.

Indeed, consider the set A of all functions g defined on subsets of D such that $g(x) \in F(x)$. We will let $\mathrm{dom}(g)$ denote the domain of definition of g. The set A is not empty, for if we pick a point $x_0 \in D$ and pick $y_0 \in F(x_0)$, then the function g whose domain consists of the single point x_0 and whose value $g(x_0) = y_0$ gives an element of A. Put a partial order on A by saying that $g \preceq h$ if $\mathrm{dom}(g) \subset \mathrm{dom}(h)$ and the restriction of h to dom g coincides with g. A linearly ordered subset means that we have an increasing family of domains X, with functions h defined consistently with respect to restriction. But this means that there is a function g defined on the union of these domains, $\bigcup X$ whose restriction to each X coincides with the corresponding h. This is clearly an upper bound. So A has a maximal element f. If the domain of f were not all of D we could add a single point x_0 not in the domain of f and $y_0 \in F(x_0)$ contradicting the maximality of f. QED